Oscillator Phase Noise: A 50-year Retrospective

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Abstract—**Fifty years ago emerging developments in oscillator applications led to the formation of an IEEE committee to unify time- and frequency-domain definitions of frequency stability. As a member, the author had the good fortune to participate and contribute. This paper is a personal recollection of events and impressions of the committee's 1964 IEEE-NASA Symposium, our 1966 *Proc IEEE* special issue on frequency stability (with comments on this author's oscillator-model paper), and our 1971 "Characterization of Frequency Stability" paper that was written to provide a basis for IEEE Std 1139.**

Keywords—phase noise, short-term stability, frequency stability, 1964 IEEE-NASA Symposium, history

I. INTRODUCTION

Oscillators, the sources of signals in electronic systems for time keeping, radio communications and radar, are characterized by frequency stability. Today the understanding of both noise-like and environmentally induced frequency instabilities in oscillators is both rigorous and readily applied. Information on the subject is widely accessible. Citation records show some 17,000 publications found in searches for "phase noise" (11,000) and "frequency stability" (6,000).

Today, standards such as IEEE 1139 offer concise definitions of concepts for the optimization of performance and interpretation of measurement [1]. Fifty years ago, differing applications had arisen in relative isolation. There was insufficient interaction to consolidate basic concepts and terminology. The current beneficial outcome was formally initiated then by an IEEE Standards subcommittee formed to unify time- and frequency-domain definitions of frequency stability. I was fortunate to participate and contribute, and this paper offers personal recollections and comments on a process that led to today's standards.

II. FREQUENCY STABILITY STANDARDS TODAY

A. Frequency Domain

Today almost the entire picture can be reduced to a few simple expressions and graphics [2]. Two sets of parameters are well-known components of standards. For an oscillator whose output is described as $V(t) = A \cos [\omega_0 t + \varnothing(t)]$, the power spectral density (PSD) $S_{\varnothing}(f)$ of the phase $\varnothing(t)$ due to random noise is modeled in frequency domain as a power-law sum $S_{\varnothing}(f) = \Sigma b_n f^{-n}$, where f is the Fourier frequency and n ranges from $4 > n > 0$. The exponents correspond to white phase ($n = 0$), flicker of phase ($n = 1$), white frequency ($n = 2$), flicker frequency ($n = 3$) and random walk frequency ($n = 4$). Equivalent forms are PSD of frequency $S_{\nu}(f)$ or normalized frequency $S_y(f) = (1/\nu_0^2)\, S_{\nu}(f) = (f^2/\nu_0^2)\, S_{\varnothing}(f)$.

The frequency-domain standard defines the measure of phase noise as $\mathcal{L}(f) = (1/2)\, S_{\varnothing}(f)$. $\mathcal{L}(f)$ is seen, for the common case $1 \ll |\varnothing|$ and AM « FM, to be a useful approximation to the RF spectrum. Dynamic range is typically limited in cell phones (the "near-far" issue) and Doppler radar (subclutter visibility) in that portion of the RF spectrum conforming to the small-angle and minimal-AM assumptions. $\mathcal{L}(f)$ is commonly presented as in Fig. 1, the log-log plot of $S_{\varnothing}(f)$ vs. f, which concisely reveals the power-law terms.

Phase noise from power supply, frequency modulation inputs and vibration or acoustic exposure has long been recognized as a mechanism that can potentially dominate quiescent noise performance. While this area represents a significant part of the author's own experience, and the literature is extensive, it is outside the scope of this paper.

B. Time Domain

The time-domain definition of stability over a time interval τ is a specific two-sample variance $\sigma_y^2(\tau)$ of normalized random frequency $y = (1/\omega_0)d\varnothing/dt$, known as the Allan Variance. The Allan Variance, also a power-law sum, can be derived from, and exists for, all spectral density power laws encountered in physical oscillators. This permits conversion from frequency domain measurements to time domain. A modified form is better suited for showing this relationship, also shown in Fig. 1, a log-log plot of $\sigma_y(\tau)$ vs. τ, whose segments correspond to the exponents of the PSD $S_{\varnothing}(f)$. It can be seen that over longer times, the low-frequency terms of phase PSD dominate. [*after NIST Pub 1065*]

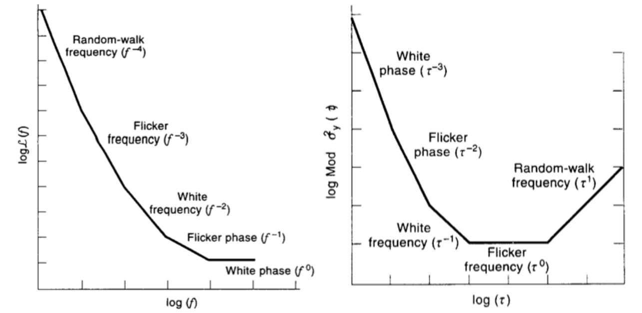

Fig. 1. Log-log plots of $S_{\varnothing}(f)$ vs. f and $\sigma_y(\tau)$ vs. τ, showing segments

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III. HISTORY

A. Frequency Stability before 1960

This enlightened state of affairs did not exist fifty years ago. Before 1940, physics and radio made use of frequency standards such as the WWV stations that employed the best quartz crystal techniques of the day. Instrumentation followed to provide portable metrology.

The need for large quantities of quartz crystals arose at the time of WWII with the introduction of channelized radio communications for mobile and airborne warfare. Problems of volume manufacturing and aging were identified and resolved to the necessary degree [3].

B. Frequency Stability at the Beginning of the 1960's

After the war's end, promising new developments such as semiconductor devices, quantum-physics frequency and timekeeping devices, television, mobile radio communication, microwave Doppler radar and even space rocketry were ripe for development. By the 1960's, applications of stable oscillators fell into two broad classes.

One, precision time and frequency standards and metrology, found the expression and measurement of frequency instabilities most natural in time-domain terms. The other, multi-signal systems such as Doppler radar and radio communications, with their dynamic range limitations due to spectrum, turned to frequency-domain definition and measurement as more applicable. Many systems were newly exposed to more stringent environments, as well.

The annual Symposium on Frequency Control, sponsored for many years by the U.S. Army, and then by the IEEE, served well as a common forum. In electronic circuits and quantum devices, the frequency-domain papers of this period focused on the RF spectrum of the oscillator itself, or on linewidth, rather than the power spectrum of phase [4]-[7]. The result was that the predicted theoretical spectra were not in complete agreement with complex observed spectra [8].

The tools appropriate for each application (long-term and time-domain in time-keeping and frequency standards, short-term and frequency-domain in radar, and a combination of both in communications) evolved along divergent paths as if in something of a guild system. Then the revolutionary developments of the transistor, the integrated circuit, digital computing and communications techniques and even the large space rocket gave rise to new requirements.

IV. IEEE SUBCOMMITTEE 14.7

A. The Formation of the Subcommittee

The emergence of communications and ranging techniques in the space program created a need for understanding and advances in both time and frequency domain concepts, and a means to convert from one to the other. These requirements stemmed from the rise of digital modulation techniques, as well as ranging. This created a new constituency whose unique issues that were not addressed by the existing communities.

At this point, early in the Apollo, satellite and planetary exploration programs, it became apparent to the several communities that they were experiencing the parable of the blind men and the elephant, and that some effort was required to pool the independent reservoirs of knowledge. The urgent NASA interest in finding common terminology for oscillator and system specifications found fertile ground in the IEEE, and a subcommittee of Technical Committee Standards 14 — Piezoelectric and Ferroelectric Crystals was proposed to explore a cross-discipline standard.

In response to this impetus, the Technical Subcommittee, Standards 14.7 — Frequency Stability was established to serve as a focal point for information in the field. The ultimate aim of the Subcommittee was an IEEE standard on the definition and measurement of both short-term and long-term frequency stability. It was at this time that I was fortunate to receive an invitation to join the Subcommittee through a mentor and sponsor, W. K. Saunders, who was familiar with my prior publications and my work on Doppler radar at Hughes Aircraft Co. My early contact with oscillator noise came as solid-state signal sources began to be applied to the radars that had been under development since the days of the MIT Radiation Laboratory. I was initiated into the phase-noise requirements of airborne Doppler radar as I applied the nonlinear frequency multipliers of my graduate theses.

Subcommittee 14.7 attracted members from the full range of user and instrument communities, and in discussion it was realized that a cross-specialty symposium would be very useful for exchange of viewpoints and techniques that would promote convergence. The committee sensed that the separate use of frequency-domain and time-domain definitions stood in the way of development of a common standard. We hoped to find a common language to discuss frequency stability, one that could be understood by everyone in the discipline. The Subcommittee focused first on the short-term frequency stability regime, in which there were greatest differences of viewpoint among the multiple user communities.

B. IEEE-NASA Symposium on Short-term Frequency Stability

The first step of the program to craft a standard that would define frequency stability was to understand and meld the frequency- and time-domain descriptions of phase instability to a degree that was mutually accepted, and that permitted analysis and optimization. To promote focused interchange as an extension of its own discussions, the Subcommittee acting as a program committee sponsored the November 1964 IEEE/NASA Conference on Short Term Frequency Stability. That conference, with some 350 attendees, was an opportunity for the cross-fertilization of ideas, and featured papers on all aspects of generation, application and measurement of short-term frequency stability. The Symposium proceedings give an insight into ripening questions and authoritative answers [9].

Of particular interest to me is the record of four panel discussions, led by prominent scientists and engineers of the time. The tension between rigorous theory and practical experiment came out often, as did the concern that adoption of a single standard would leave a portion of the community without the tools for its specific applications. Also evident was the full range of individual specialization and experience, and even of personality types.

Specific questions were raised about the uncertainties of the correspondence between near-carrier linewidth and RF spectrum and an underlying spectral density of phase or frequency. The conundrum of the origin of flicker noise with its lack of convergence of integrals at zero frequency received substantial questioning and discussion. Additionally, the subject arose whether higher order effects or amplitude noise were adequately recognized, and it was concluded that experimental evidence supported the idea that these were not significant in then-current applications.

One of my own curiosities on reviewing the Symposium proceedings from this remove was to identify how and where certain key concepts were conclusively identified in the papers. For example, there was substantial discussion of power-law descriptions of PSD of phase and its relationship to RF spectrum, but my review found no graphic that explicitly showed the segments. The issue of converting from frequency to time domain was explored, but not resolved at that early time. A number of authors noted flicker noise in amplifiers and other physical devices. There were several efforts to relate the output spectrum of an oscillator to the characteristics of the resonator feedback network and the active device, but the full connection between the amplifier PSD, resonator and amplifier parameters and the output PSD remained to be clarified.

C. 1966 Proceedings Special Issue on Frequency Stability

With the success of the 1964 Symposium, in order to consolidate the gains and promote further exchange of information, the Subcommittee was invited to serve as editorial committee of the February 1966 Special Issue on Frequency Stability of the Proceedings of the IEEE [10]. This issue attracted many submittals, including updated papers from the 1964 Symposium, including several by committee members who were also among the most active in the field.

We were most pleased to receive a paper by D. W. Allan that settled issues of time domain definitions and techniques, as well as showing how to convert from frequency domain definitions [11]. By this time it had become accepted that spectral density of phase or frequency, rather than RF spectrum or linewidth, was the more fundamental frequency-domain measure of short-term stability. PSD could be directly related both to the time domain definition that became identified as the Allan Variance, and within limits it predicted the RF spectrum. The IEEE 1139 standard now applies the small-angle limitation in reverse, such that the RF spectrum is *defined* as half the PSD of phase *except* where the small-angle condition is not met.

In preparation for this paper, I revisited the special issue, again curious to find when and where key points became clear. Although there were many instances of power-law spectra, I still found no example of the now-accepted multi-segment PSD of phase. Papers that dealt with determining oscillator output PSD from input PSD were restricted to a subset of the overall question. Questions remained regarding flicker noise, nonlinearities, the interrelation of PSD of phase and RF spectrum, and AM noise. A number of papers dealt with flicker noise, including one that specifically mentioned flicker noise in resonators Papers on oscillator-multipliers suggested a choice of higher oscillator frequency because of the multiplication of modulation index. The radar community, responding to vibration problems, was adopting the ribbon-mounted quartz crystal developed at the Bell Telephone Laboratories.

D. Model of Feedback Oscillator Phase Noise Spectrum

In our final deliberations to settle the contents of the special issue, it seemed to me that we had not received a paper on frequency-domain techniques that was as clear as the Allan paper was on time-domain issues. I thought I could see a way to create such a paper.

A paper selected for our special issue showed that, subject only to conditions that were typically met in oscillators, for a nonlinear circuit driven by a periodic input, the AM and PM noise could be treated as strictly linear and stochastic, and thus could be described in terms of spectral densities [12]. Its author and I had been graduate students together at MIT, sharing a thesis advisor. Adding to my own background in nonlinear circuits, this encouraged my interest in synthesizing a simple quasi-linear model of oscillator noise behavior.

At this point it seemed enough was known to assemble a model that used the power-law forms of PSD, with graphics to provide additional clarity. The input and output PSD could be related by a transfer function reflecting the key parameters of the active element and resonator. This transfer function would be an extension of those of Edson and Baghdady to include the model for white noise outside the resonator bandwidth from my own Doppler-radar papers. I felt that a quasi-linear model of phase noise as a small perturbation of the oscillator steady-state signal, even in a nonlinear oscillator, would have broad applicability. Since this was very late in the editing process, I was encouraged by the committee to submit a concise paper that would be published as correspondence, since that section was held open for late submittals

That was the origin of my 1966 paper on the oscillator noise model [13]. Looking back, I am satisfied with what I was able to shoehorn into two pages, submitted after discussion with colleagues by Dec. 29, barely a month before our publication date. Proceedings correspondence was not archived, so for a number of years the paper remained obscure except to insiders. During that same time I was fully occupied with founding and managing a new company, so the paper led something of a life of its own. I am pleased to find its continuing utility has raised it to the most cited paper in the "phase noise" category.

In the intervening fifty years there have been advances in the clarity with which the concepts could be expressed, and the model has been extended to new frequency-determining and active elements. Much later, novel requirements and solutions would arise from the emergence of the integrated circuits that would completely reshape what was possible in electronics, and would require new approaches. Many of the questions about nonlinearity and RF and *1/f* spectrum have been resolved, through physical argument or mathematical rigor.

V. COMMENTS ON THE PHASE NOISE MODEL PAPER

A. Spectral Models of Phase Variations

As was already the practice, the oscillator output was taken as $v(t) = A \cos [\omega_0 t + \varnothing(t)]$, where $\varnothing(t)$ is treated as a zero-mean stationary random process. In Symposium papers and discussions among attendees and committee members, it had been concluded that power spectral density of phase $S_{\varnothing}(\omega_m)$, or its equivalent, the PSD of frequency $S_{\nu}(\omega_m) = \omega_m^2 S_{\varnothing}(\omega_m)$, represented the most suitable definition of phase noise instabilities (as opposed to RF spectrum or linewidth, from which one could not necessarily determine a unique spectrum of phase).[1] Here ω_m was taken as the Fourier frequency associated with the noise-like variations in $\varnothing(t)$. Subject to the limitation AM « FM and to the small angle approximation $<\varnothing>^2 << 1$, the normalized RF power spectrum was related by a constant to $S_{\varnothing}(\omega_m)$.

B. Power Spectral Density of Oscillator Internal Phase Noise

For the VHF overtone crystal feedback oscillator used as the basis for the model, the spectrum $S_{\Delta\theta}(\omega_m)$ of the input phase uncertainty $\Delta\theta(t)$ was taken to have two components, flicker *1/f* modulation and additive white noise around the oscillator frequency, including "noise at other frequencies mixed into the pass band of interest by nonlinearities."

The spectral density of input phase due to additive white noise was known from modulation theory to be the ratio of noise power to signal power. For a feedback oscillator with an *effective* noise figure F (giving effect to nonlinear mixing), the *two-sided* input spectrum $S_{\Delta\theta}(\omega_m) = 2FKT/P_S$, where P_S was taken to be the signal level at the input of the oscillator active element. The factor of 2 was deleted in subsequent papers in a change to one-sided spectra.

The second component of input phase spectrum was seen to be parameter variations that modulate the internal phase at video or baseband rates. This modulation, whose PSD typically varies as *1/f*, impresses its effect on the oscillator signal without any appeal to nonlinearity. This modulation component is independent of signal amplitude. The flicker variation of the resonator itself, which was beginning to be appreciated by those working with lower frequency oscillators, was not observed then in VHF crystals and was disregarded by me at that time.

A suitable expression for total spectral density of input phase errors was (and is) of the form $S_{\Delta\theta}(\omega_m) = \alpha/\omega_m + \beta$, where α is a constant determined by the magnitude of *1/f* flicker variations and $\beta = FKT/P_S$ for one-sided spectra. The already modest noise figure F of those days was raised by "corrections necessary to account for nonlinear effects, which must be present in a physical oscillator." This line of thought was expanded then in a following section on nonlinearity.

C. Relation to Oscillator Internal Noise

A key intent of the model was "to show clearly the relationship of the output spectral density of phase $S_{\varnothing}(\omega_m)$ to the known or expected noise and signal levels and resonator characteristics of the oscillator," F, P_S, Q and ω_o. The simplest model was that of a linear feedback oscillator.

To deduce from physical reasoning the transfer function from input phase spectrum to output phase spectrum, the paper considered a single-resonator feedback network of fractional bandwidth $2B/\omega_0 = 1/Q$, with Q the loaded quality factor. For small phase variations that fall within the feedback half-bandwidth $\omega_0/2Q$, a phase error at the oscillator input due to noise or parameter variations would result in a frequency error determined by the phase-frequency slope of the feedback network, $\Delta\theta = 2Q\Delta\nu/\omega_0$.

For modulation rates *large* compared to the feedback bandwidth, the feedback loop has no effect, so for this regime the output power spectral density $S_{\varnothing}(\omega_m)$ was seen to equal the input spectral density $S_{\Delta\theta}(\omega_m)$. Thus a suitable expression for the transfer function was given as $|H(\omega)|^2 = [1 + (\omega_0/2Q\omega_m)^2]$.

D. Power Spectral Density of Output Phase

The PSD of output phase was then shown to be just the product of the input spectrum and the transfer function, so $S_{\varnothing}(\omega_m) = S_{\Delta\theta}(\omega_m)|H(\omega)|^2 = [\alpha/\omega_m + FKT/P_S]\ [1 + (\omega_0/2Q\omega_m)^2]$. This could be shown simply in the graphic construction of Fig. 2 that identified the feedback bandwidth and the breakpoint of the flicker segment. Note that flicker noise is typically modulative, and hence does not vary with F or P_S.

The example given was the case in which *1/f* effects predominate only for frequencies small compared with the feedback loop bandwidth. It was noted that if flicker predominates, the breakpoints would be interchanged. The caption noted that the RF spectrum could be derived subject to the limitations (small-angle, AM « FM) in the text. One-sided rather than two-sided spectra became the norm at a later time, thus resolving a factor of two in the white noise level. A significant point is that this figure and the measurement made graphically explicit the power-law segments of oscillator PSD.

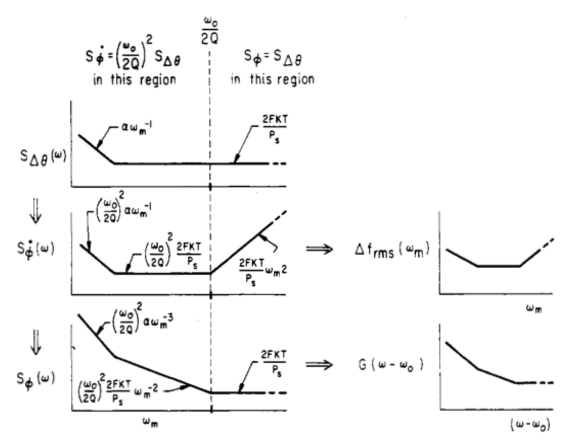

Fig. 2. Graphical derivation of output $S_{\varnothing}(\omega_m)$ from $S_{\Delta\theta}(\omega_m)\ |H(\omega)|^2$

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[1] As a note, the original paper used a dot over the symbol ø to denote the time derivative ("Newton's notation"). Recalling the frustration of chasing dots pasted to the galley proof, the author heartily approves of the modern use of ν or Ω for frequency. This problem also plagued other papers in the issue.

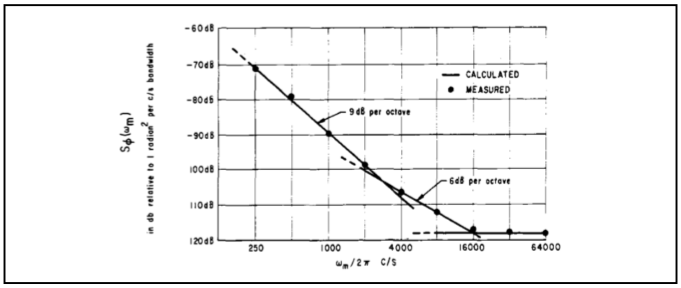

Fig. 3. Output PSD explicitly showing multiple power law segments

E. *Output PSD Experimental Verification*

A measurement was presented to validate the theoretical model. The model and data are compared in Fig. 3, reproduced from the original paper. The agreement was reassuring.

F. *Video Frequency Range of Interest*

Space systems and Doppler radar were of particular interest to the author. Space data links used narrow bandwidth, and so the low Fourier frequencies of the flicker segment were seen as critical. Radar requirements ranged up to 100 kHz. The name "Hertz" as the unit of frequency had just been adopted in 1965, and was not yet in common use.

G. *Choice of Oscillator Frequency*

Frequency multipliers were known to increase modulation index by a factor equal to the multiplication ratio N, so PSD is increased by N^2. Our work was at 10 GHz. For a given output frequency, the choice of oscillator frequency is significant.

The graphical construction in Fig. 4, alluded to but not included in the paper, shows that a higher oscillator frequency yields lower noise for Fourier frequencies above the resonator bandwidth. From comparisons such as this, it was also seen that the most favorable PSD segments of oscillators could be combined by use of phase lock loops in synthesizers.

H. *1/f noise in the active element*

Since *1/f* variations and nonlinearity compromised the achievable PSD, it was suggested that AGC oscillators with large-area high-power transistors could provide simultaneous improvements in flicker and nonlinear effects. Later it was found that bipolar devices were better in this aspect.

It was pointed out that output PSD could be modified by subsequent bandlimiting filtering and by the noise of following amplifiers. Last, the potential was noted for coupling the signal directly from the resonator to filter the white noise component.

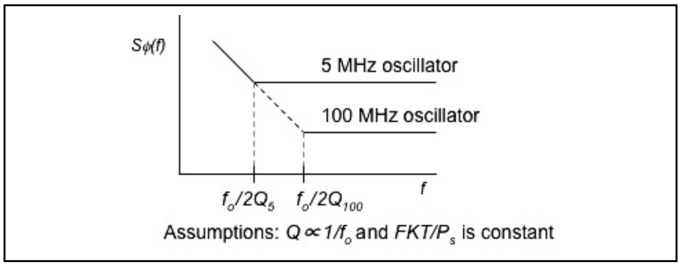

Fig. 4. Graphical construction for choice of VHF oscillator

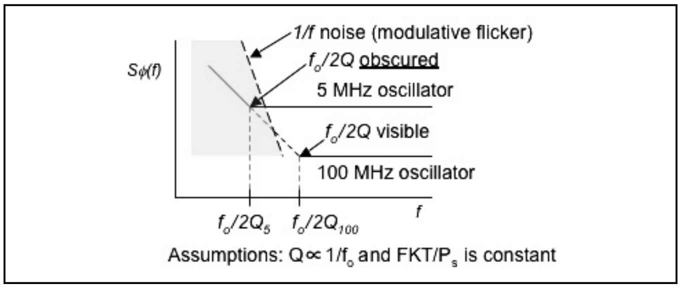

Fig. 5. 1/f noise obscures $f_0/2Q$ for high-Q or low-power oscillator

It was noted that, "For a high Q oscillator, *1/f* effects in $S_{\Delta\theta}$ can predominate out to a modulation rate exceeding $(\omega_0/2Q)$" in which case the slope transition at the resonator half bandwidth is obscured, as shown in Fig. 5. Because flicker modulation does not vary with signal power as both the f^0 and f^{-2} segments do, it can also obscure the resonator parameters for low oscillator power.

I. *Nonlinear Effects*

To raise the estimate of noise figure F above the status of a fitting factor, I had estimated an added 4 dB above the published small-signal value "to account for *nonlinear mixing* of noise at third harmonic and higher frequencies." This is shown schematically in Fig. 6. The paper was directed at the VHF oscillator type then typical of Doppler radar applications I worked with, with the expectation that the result would have more general applicability.

VI. "CHARACTERIZATION OF FREQUENCY STABILITY"

By the end of the 1960's, it was felt that sufficient progress had been made that we could prepare a paper to summarize understanding that would underlie a future standard. Produced in the days before the Internet and email, this involved numerous discussions and correspondence among the ten authors. Despite the complexity of responding to all viewpoints, we deemed the result to be a useful step forward, and it was published in 1971 [14].

VII. LIMITS AND EXTENSIONS OF THE SIMPLE MODEL

Over time, the following questions have properly been raised about the limits of applicability of the simple model:

- Does F reflect nonlinearity and circuit impedances?
- Extension to other frequency-determining elements?
- Is flicker of resonator recognized?
- Near-carrier limits of conversion from PSD to RF?
- Is AM to PM conversion recognized?

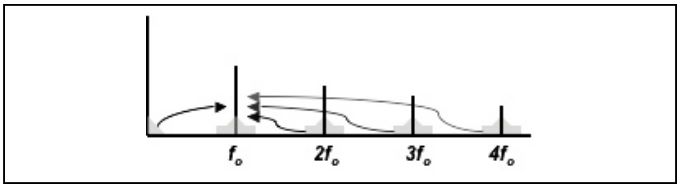

Fig. 6. Nonlinear mixing of noise from harmonics, plus *1/f* modulation

A. *Nonlinearity*

In time, strongly nonlinear oscillators arose from the proliferation of semiconductor integration. These were treated by quite different fundamentally nonlinear analyses in more recent papers, in one case by close colleagues at Stanford [15]-[16]. However, measurements of high-Q oscillators continue to confirm the persistent utility of the simple model [17].

B. *Extended frequency determining elements*

A range of new frequency determining elements has arisen over time, including delay lines, bandpass filters and multiple resonators. In many cases, the simple model is extended by determining the phase slope from group delay τ rather than Q.

C. *Flicker in Active Elements and Resonators*

As proposed in the phase noise model paper, the reduction of flicker noise by feedback and choice of active element resulted in substantial improvement in oscillator stability in a relatively short time [18]. As can been seen from a graphical construction, this effect would be much greater in high-Q HF oscillators, as opposed to lower-Q VHF overtone oscillators [19]. Flicker noise in the resonator itself had been suggested in a 1966 paper [20]. Investigations confirmed the significance of flicker in resonators [21]. This noise source was less observed in VHF crystals and was disregarded by me in the simple model, which has been extended to recognize this [22].

The nature and effect of flicker noise has been the subject of substantial attention in subsequent years. A confounding problem was the infinity at zero frequency for PSD rising as $1/f^N$. It has been suggested that finite bandwidth and measurement time create the equivalent of a bandpass filter that acts to truncate the PSD [23].

D. *Near-carrier Large Modulation Index*

A related issue is the regime of large modulation index where the small-angle assumption is not valid, in which the spectral density of phase grows without limit, typically very near the carrier. This issue has been termed the "infrared catastrophe" by allusion to the ultraviolet catastrophe of pre-quantum radiation physics. Papers before the 1960's modeled the RF spectrum of an oscillator frequency modulated by noise with components down to zero frequency. More recent papers confirm mathematically that the output power of the modulated oscillator remained constant as expected, and that the close-in RF spectrum shape is Lorentzian or Gaussian [24].

E. *AM-PM Conversion*

The effect of AM-PM conversion remains a concern that must be considered. Oscillators generally meet the criterion AM « PM, and experiment has shown it not to be a primary issue in many systems of interest. By modulation theory, equal RF sidebands confirm that one or the other form dominates. From experience, this is typically phase noise in an oscillator with limiting or frequency multiplication.

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