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Project Report
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A Comparison of Immunity to Garbling for Three Candidate Modulation Schemes for DABS

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1. INTRODUCTION

One of the major issues in the development of a Discrete Address Beacon System (DABS) for surveillance of air traffic is that of compatibility with the present surveillance system, the Air Traffic Control Radar Beacon System (ATCRBS) which operates on separate uplink and downlink frequencies of 1030 and 1090 MHz, respectively. Since the aircraft population will take years to convert from ATCRBS to DABS transponders, there will be a significant period of time in which DABS and ATCRBS will have to co-exist. It is desirable to build ATCRBS capability into both airborne and ground DABS equipment to accommodate a gradual evolution to a DABS. It is thus reasonable to investigate the possibility of DABS use of the ATCRBS frequencies in order to allow maximum sharing of both airborne and ground equipment for design economy.

The problem that is investigated in this report is the sensitivity of candidate DABS modulation systems to interference, either from ATCRBS or DABS transmissions, or as is likely to arise from multipath reflections. We are particularly interested in assessing the question of sensitivity of DABS operation on ATCRBS frequencies. Hence, the performance of DABS candidate modulations are analyzed for three types of environment:

(1) Additive gaussian noise.

(2) DABS-like interfering waveforms with gaussian noise.

(3) ATCRBS interference with gaussian noise.

The three modulation schemes that are analyzed are pulse amplitude modulation on-off keyed (PAM), differential phase shift keying (DPSK), and frequency shift keying (FSK). The performance of these three modulation systems is compared on the basis of probability of error per bit, P_e/bit as a function of signal-to-noise and interference-to-signal ratios.

Ultimately, it is the reliability of coded message blocks which is of interest, but the calculation of P_e/bit is a necessary first step in obtaining message reliability. In this report, analytical expressions are obtained for the maximum likelihood demodulations [1] followed by the optimum decision strategy (the minimum error rate decision strategy). A familiarity with the papers by Arthurs and Dym [2] and Stein [3] is assummed.

At this point, we introduce the following useful identity and notation:

IDENTITY

$$\cos(\omega t + \alpha) + \rho \cos(\omega t + \beta) = \sqrt{1 + 2\rho \cos\theta + \rho^2} \cos(\omega t + \phi) \quad , \tag{1}$$

where

$$\theta = \beta - \alpha \quad \text{and} \quad \phi = \tan^{-1} \frac{\sin \alpha + \rho \sin \beta}{\cos \alpha + \rho \cos \beta} \quad . \tag{2}$$

If β and α are uniformly distributed $-\pi$ to π, then θ and ϕ are also uniformly distributed $-\pi$ to π.

NOTATION

E is the signal energy

ρ^2 is the ratio of the interfering pulse energy to the signal energy

T is the pulse duration

$N_0/2$ is the double sided white noise spectral density

$$R_+(\theta) = \sqrt{\frac{2E}{N_0}(1 + 2\rho\cos\theta + \rho^2)} \quad , \tag{3}$$

and

$$R_-(\theta) = \sqrt{\frac{2E}{N_0}(1 - 2\rho\cos\theta + \rho^2)} \quad . \tag{4}$$

2. INCOHERENT PAM

The first modulation scheme considered is incoherent PAM. If we assume that the interfering pulse completely overlaps the information pulse, then the waveform at the input to the receiver is, for a peak power limited signal,

$$\begin{aligned} r(t) &= \sqrt{\frac{2E}{T}}\,[c_1 \cos(\omega t + \alpha) + c_2\,\rho\cos(\omega t + \beta)] + n(t) \\ &= \sqrt{\frac{2E}{T}(c_1^2 + 2c_1 c_2\,\rho\cos\theta + c_2^2\rho^2)}\ \cos(\omega t + \phi) + n(t) \ , \end{aligned} \tag{5}$$

where c_1 and c_2 are each either 0 or 1.

This received signal is demodulated using the maximum likelihood estimate [1] to obtain

$$x = \int_0^T r(t)\sqrt{\frac{2}{T}}\cos\omega t\, dt \quad ,$$

$$= \sqrt{E\left[c_1^{\,2} + 2c_1c_2\,\rho\cos\theta + c_2^{\,2}\,\rho^2\right]}\quad \cos\phi + n_1 \quad , \tag{6a}$$

and

$$y = \int_0^T r(t)\sqrt{\frac{2}{T}}\sin\omega t\, dt$$

$$= -\sqrt{E[c_1^{\,2} + 2\,c_1c_2\,\rho\cos\theta + c_2^{\,2}\,\rho^2]}\ \sin\phi + n_2 \ , \tag{6b}$$

where n_1 and n_2 are independent zero mean gaussian random variables with variance $N_0/2$ so that (see Ref. 2, p. 354 Eq. (93))

$$p(x,y\big|\theta) = \frac{1}{2\pi(N_0/2)}\, e^{-\frac{1}{2}\left\{\frac{x^2 + y^2 + E[c_1^{\,2} + 2c_1c_2\,\rho\cos\theta + c_2^{\,2}\,\rho^2]}{N_0/2}\right\}}$$

$$I_0\left(\frac{\sqrt{x^2 + y^2}\sqrt{E[c_1^{\,2} + 2c_1c_2\,\rho\cos\theta + c_2^{\,2}\,\rho^2]}}{N_0/2}\right) \quad . \tag{7}$$

Using the change of variables

$$x = v\sqrt{\frac{N_0}{2}}\cos\gamma,\ y = v\sqrt{\frac{N_0}{2}}\sin\gamma \quad ,$$

and the fact that θ is uniformly distributed then we obtain

$$p(v,\gamma,\theta) = \frac{1}{4\pi^2}\, v\, e^{-\frac{1}{2}\left\{v^2 + \frac{2E}{N_0}\left[c_1^{\,2} + 2c_1c_2\,\rho\cos\theta + c_2^{\,2}\rho^2\right]\right\}}$$

$$\times\, I_0\left(v\sqrt{\frac{2E}{N_0}\left(c_1^{\,2} + 2c_1c_2\,\rho\cos\theta + c_2^{\,2}\rho^2\right)}\right). \tag{8}$$

$p(v|\rho)$ is obtained from (8) simply by integrating from $-\pi$ to π on γ and θ. The optimum receiver is determined from the likelihood ratio

$$\Lambda(v|\rho) = \frac{P(v|H_1,\rho)}{P(v|H_0,\rho)} ,$$

where H_0 is the hypothesis of a space ($c_1 = 0$) and H_1 is the hypothesis of a mark ($c_1 = 1$). Each hypothesis is equally likely. The likelihood ratio is therefore

$$\Lambda(v|\rho) = \frac{e^{-\frac{E}{N_0}}\, I_0\left(v\sqrt{\frac{2E}{N_0}}\right) P_{20} + e^{-\frac{E}{N_0}(1+\rho^2)}\, \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-\frac{2E}{N_0}\rho\cos\theta}\, I_0\left(vR_+'(\theta)\right) d\theta\, P_{21}}{P_{20} + e^{-\rho\frac{E}{N_0}}\, I_0\left(v\sqrt{\frac{2E}{N_0}}\right) P_{21}} , \tag{9}$$

where $R_+(\theta)$ has been defined in (3), P_{20} is the probability that $c_2=0$, and $P_{21}=1-P_{20}$. To determine the optimum threshold level for v, we solve

$$\Lambda(v|\rho) = 1 ,$$

or equivalently solve for the roots, v_0, of the equation

$$f(v) = e^{-\frac{E}{N_0}} \quad I_0\left(v\sqrt{\frac{2E}{N_0}}\right)P_{20}$$

$$+ e^{-\frac{E}{N_0}(1+\rho^2)} \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-\frac{2E}{N_0}\rho \cos\theta} \quad I_0\ (v\ R_+(\theta))d\theta\ P_{21}$$

$$- P_{20} - e^{-\rho^2 \frac{E}{N_0}} I_0\left(v\rho\sqrt{\frac{2E}{N_0}}\right)P_{21} = 0 \quad . \tag{10}$$

v_0 is therefore the optimum threshold level for the receiver in an interference environment. When $P_{20}=1$ (i.e., $P_{21}=0$) then an excellent approximation to the solution of (10) is (Schwartz, Bennett, and Stein [4] Eq. 7-4-14)

$$v_0 \approx \sqrt{\frac{E}{2N_0} + 1} \quad . \tag{11}$$

When there is interference ($P_{21}\neq 0$) then (11) no longer approximates the optimum threshold level but P_e is far more sensitive to the parameter ρ than to v_0.

It is useful to introduce Marcum's Q function [5] defined by the equation

$$Q(\lambda,\beta) \doteq \int_{\beta}^{\infty} v\ e^{-\frac{1}{2}(v^2+\lambda^2)}\ I_0\ (\lambda\ v)\ \ dv \quad . \tag{12}$$

The P_e for the four possible cases are determined as follows:

(i) $c_1 = 0,\ c_2 = 0$

$$p_i(v) = v\ e^{-v^2/2} \tag{13}$$

$$P_{e,i} = \int_{v_0}^{\infty} p_i(v)dv = e^{-v_0^{\,2}/2} = Q(0,v_0) \tag{14}$$

(ii) $c_1 = 1,\ c_2 = 0$

$$p_{ii}(v) = v\ e^{-\frac{1}{2}\left(v^2 + \frac{2E}{N_0}\right)}\quad I_0\left(v\sqrt{\frac{2E}{N_0}}\right) \tag{15}$$

$$P_{e,ii} = \int_{0}^{v_0} p_{ii}(v)dv = 1 - Q\left(\sqrt{\frac{2E}{N_0}},\ v_0\right) \tag{16}$$

(iii) $c_1 = 0,\ c_2 = 1$

$$p_{iii}(v) = v\ e^{-\frac{1}{2}\left(v^2 + \rho^2\frac{2E}{N_0}\right)}\quad I_0\left(v\ \rho\ \sqrt{\frac{2E}{N_0}}\right) \tag{17}$$

$$P_{e,iii} = \int_{v_0}^{\infty} p_{iii}(v)\ dv = Q\left(\rho\sqrt{\frac{2E}{N_0}},\ v_o\right) \tag{18}$$

and finally

(iv) $c_1 = 1,\ c_2 = 1$

$$p_{iv}(v) = \frac{1}{2\pi} \int_{-\pi}^{\pi} v\, e^{-\frac{1}{2}[v^2 + R_+(\theta)]} I_0(v\, R_+(\theta))\, d\theta \ , \tag{19}$$

$$P_{e,iv} = \int_0^{v_0} p_{iv}(v)dv = 1 - \frac{1}{2\pi} \int_{-\pi}^{\pi} Q(R_+(\theta), v_0)\, d\theta \ . \tag{20}$$

In the case of DABS-like interference due to multipath or a second DABS interrogator, the probabilities of each of the above cases is equally likely, so

$$P_e/\text{bit} = \frac{1}{4}\left[e^{-\frac{v_0^2}{2}} + 1 - Q\left(\sqrt{\frac{2E}{N_0}},\ v_0\right) + Q\left(\rho\sqrt{\frac{2E}{N_0}}, v_0\right) + 1 - \frac{1}{2\pi} \int_{-\pi}^{\pi} Q(R_+(\theta),\ v_0)\, d\theta\right]. \tag{21}$$

For ATCRBS interference, we shall consider a slightly different quantity, namely, the average P_e/garbled bit, P_{eg}. This is just the average of case (iii) and case (iv),

$$P_{eg} = \frac{1}{2}\left[Q\left(\rho\sqrt{\frac{2E}{N_0}},\ v_0\right) + 1 - \frac{1}{2\pi} \int_{-\pi}^{\pi} Q(R_+(\theta),\ v_0)\, d\theta\right] \ . \tag{22}$$

3. DPSK

The k^{th} received waveform for DPSK modulation, assuming complete overlap of the interfering signal, is of the form

$$r_k(t) = \sqrt{\frac{2E}{T}}\,[\cos(\omega t + \alpha_k) + c_k\,\rho\cos(\omega t + \beta_k)] + n(t)$$

$$= \sqrt{\frac{2E}{T}\left[1 + 2c_k\,\rho\cos\theta_k + c_k^2\,\rho^2\right]}\cos(\omega t + \phi_k) + n(t)\,, \qquad (23)$$

where $c_k = 0$ or 1. The maximum-likelihood estimate demodulation [1] yields from (6a) and (6b)

$$x_k = \sqrt{E\left(1 + 2\,c_k\,\cos\theta_k + c_k^2\,\rho^2\right)}\cos\phi_k + n_{xk}\,, \qquad (24a)$$

$$y_k = -\sqrt{E\left(1 + 2\,c_k\,\cos\theta_k + c_k^2\,\rho^2\right)}\sin\ {}_k + n_{yk}\,, \qquad (24b)$$

where n_{xk} and n_{yk} are independent zero mean gaussian random variables with variance $N_0/2$. The optimum decision criterion, as determined in Appendix A, is simply to compare the angles of (x_{k-1}, y_{k-1}) and (x_k, y_k) and if their difference is less than $\pi/2$, decide that the two transmitted phases, α_{k-1} and α_k, were the same; otherwise, decide that α_{k-1} and α_k differ by π radians.

From a symmetry argument, it can be seen that the P_e/bit can be calculated from the particular case where $\alpha_{k-1} = \alpha_k$. There are three situations which must be considered:

(i) $c_{k-1} = c_k = 1$ and $\beta_{k-1} = \beta_k$

(ii) $c_{k-1} = c_k = 1$ and $\beta_{k-1} \neq \beta_k$

(iii) $c_{k-1} = 0$, $c_k = 1$ or $c_{k-1} = 1$, $c_k = 0$.

We now obtain the P_e for each as follows:

Case (i) For this case, the P_e is the same as derived in Arthurs and Dym [2] or Stein [3] except that signal-to-noise ratio is replaced by $R_+^2(\theta)$ and an averaging is taken with respect to θ (see Figure 1(a)).

$$P_{e,i} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{2} e^{-\frac{E}{N_0}(1 + 2\rho\cos\theta + \rho^2)} d\theta$$

$$= \frac{1}{2} e^{-\frac{E}{N_0}(1+\rho^2)} I_0\left(2\rho \frac{E}{N_0}\right) . \qquad (25)$$

Case (ii) In this case, the derivation of P_e is somewhat complicated since we have $\beta_{k-1} = \beta_k \pm \pi$. The pertinent figure is Figure 1(b). For a given value of ϕ_k and θ_k, the P_e is the probability that the projection of the additive gaussian noise along the reference phase axis is greater than d (see Figure 1(b)), that is,

$$P(\text{error}|\phi_k,\theta_k) = \frac{1}{\sqrt{2\pi}} \int_{R_-(\theta_k)\cos(\phi_k + \psi_1 + \psi_2)}^{\infty} e^{-x^2/2} dx , \qquad (26)$$

where

$$\psi_1 = \tan^{-1} \frac{\rho \sin\theta_k}{1 + \rho\cos\theta_k} , \qquad (27a)$$

and

$$\psi_2 \tan^{-1} \frac{\rho \sin\theta_k}{1 - \rho\cos\theta_k} . \qquad (27b)$$

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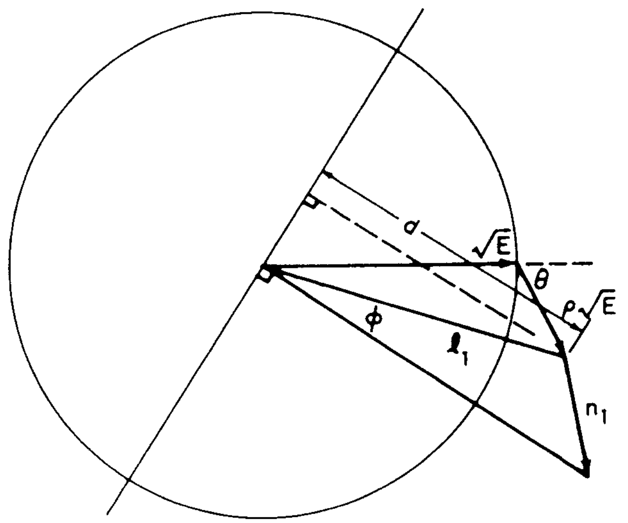
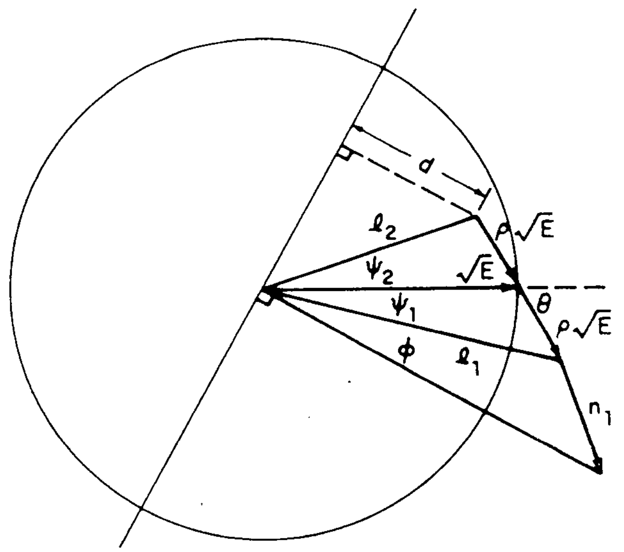


$$\ell_1 = \sqrt{E}\,[1 + 2\rho\cos\theta + \rho^2]^{1/2}$$

$$d = \sqrt{E}\,[1 + 2\rho\cos\theta + \rho^2]^{1/2}\cos\phi$$

(a) case 1



$$d = \sqrt{E}\,[1 - 2\rho\cos\theta + \rho^2]^{1/2}\cos(\phi + \psi_1 + \psi_2)$$

$$\ell_1 = \sqrt{E}\,[1 + 2\rho\cos\theta + \rho^2]^{1/2}$$

$$\ell_2 = \sqrt{E}\,[1 - 2\rho\cos\theta + \rho^2]^{1/2}$$

$$\psi_1 = \tan^{-1}\frac{\rho\sin\theta}{1 + \rho\cos\theta}$$

$$\psi_2 = \tan^{-1}\frac{\rho\sin\theta}{1 - \rho\cos\theta}$$

(b) case 2

Fig. 1. DPSK, signal with energy E, jamming with energy ρ^2E.

Rather than trying to simplify this equation, we can turn to the results of Stein [3,4] where he derives the error expression which corresponds to

$$P(\text{error}|\theta_k) = \frac{1}{2}\left[1 - Q(\sqrt{b}, \sqrt{a}) + Q(\sqrt{a}, \sqrt{b})\right] , \tag{28}$$

where

$$\begin{Bmatrix} a \\ b \end{Bmatrix} = \frac{1}{2}\left\{\frac{R_+^2(\theta_k) + R_-^2(\theta_k)}{2} \mp R_+(\theta_k)\, R_-(\theta_k)\, \cos(\psi_1 + \psi_2)\right\} , \tag{29}$$

where the minus sign corresponds to a and the plus to b. We obtain for the cosine

$$\cos(\psi_1 + \psi_2) = \frac{R_+^2(\theta_k) + R_-^2(\theta_k) - 4\rho^2 E/N_o}{2R_+(\theta_k)R_-(\theta_k)} , \tag{30}$$

which yields

$$a = \frac{2\rho^2 E}{N_0} \quad \text{and} \quad b = \frac{2E}{N_0} . \tag{31}$$

The $P(\text{error}|\theta_k)$ is independent of θ_k, so $P_{e,ii}$ becomes

$$P_{e,ii} = \frac{1}{2}\left[1 - Q\left(\sqrt{\frac{2E}{N_0}}, \rho\sqrt{\frac{2E}{N_0}}\right) + Q\left(\rho\sqrt{\frac{2E}{N_0}}, \sqrt{\frac{2E}{N_0}}\right)\right] , \tag{32}$$

which together with (25) yields for DABS-like interference

$$P_e = \frac{1}{4}\left[1 - Q\left(\sqrt{\frac{2E}{N_0}}, \rho\sqrt{\frac{2E}{N_0}}\right) + Q\left(\rho\sqrt{\frac{2E}{N_0}}, \sqrt{\frac{2E}{N_0}}\right) + e^{-\frac{E}{N_0}(1+\rho^2)} I_0\left(2\rho\frac{E}{N_0}\right)\right]. \quad (33)$$

Case (iii) Since it is irrelevent which pulse is used for the reference signal and which for the information signal, we can assume that $c_k = 0$, and $c_{k-1} = 1$. For this situation

$$\psi_2 = 0 \qquad \text{and} \qquad \cos\psi_1 = \frac{\sqrt{E}(1 + \rho\cos\theta_k)}{R_+(\theta_k)},$$

so that a and b become

$$a = \frac{E^2}{2N_0} \qquad \text{and} \qquad b = \frac{E(4 + 4\rho\cos\theta_k + \rho^2)}{2N_0},$$

yielding

$$P_{e,iii} = \frac{1}{2}\left[1 - \frac{1}{2\pi}\int_{-\pi}^{\pi} Q\left(\sqrt{\frac{E(4 + 4\rho\cos\theta + \rho^2)}{2N_0}}, \rho\sqrt{\frac{E}{2N_0}}\right) d\theta + \frac{1}{2\pi}\int_{-\pi}^{\pi} Q\left(\rho\sqrt{\frac{E}{2N_0}}, \sqrt{\frac{E(4 + 4\rho\cos\theta + \rho^2)}{2N_0}}\right) d\theta\right]. \quad (34)$$

The optimum decision criterion derived in Appendix A is not optimum for ATCRBS interference. In the case of two adjacent pulses garbled by a single ATCRBS pulse, we always have $\beta_{k-1} = \beta_k$ whereas for the situation in Appendix A, this is true only half the time while $\beta_{k-1} = \beta_k \pm \pi$ is true the other half.

4. INCOHERENT FSK

Third and finally, we consider incoherent FSK. We must differentiate between DABS-like interference and ATCRBS interference, since for the former the interfering signal will be at one of the two pertinent FSK frequencies while for the latter it will be at the ATCRBS frequency. We consider first DABS-like interference. The received signal is of the form

$$r_k(t) = \sqrt{\frac{2E}{T}}\,[\cos(\omega_k t + \alpha_k) + \rho\, c \cos(\omega_1 t + \beta) + \rho(1 - c) \cos(\omega_2 t + \beta)] \qquad k = 1,2 \qquad (35)$$

where c = 0 on 1 depending on whether the interfering signal is of frequency ω_2 or ω_1. We may assume for purposes of calculating the P_e that k = 1. The maximum likelihood estimate demodulator [1] yields from (6a) and (6b)

$$x_1 = \sqrt{E(1 + 2\, c\, \rho \cos\theta + c^2 \rho^2)}\, \cos\phi_1 + n_{x1} \qquad (36a)$$

$$y_1 = -\sqrt{E(1 + 2\, c\, \rho \cos\theta + c^2 \rho^2)}\, \sin\phi_1 + n_{y1} \qquad (36b)$$

and

$$x_2 = \rho(1 - c) \sqrt{E} \cos \phi_2 + n_{x2} , \tag{37a}$$

$$y_2 = -\rho(1 - c) \sqrt{E} \sin \phi_2 + n_{y2} , \tag{37b}$$

where it has been assumed that ω_1 and ω_2 are chosen so that no crosstalk exists and that n_{xk} and n_{yk} are independent zero mean gaussian random variables with variance $N_0/2$. If we let

$$x_k = v_k \sqrt{\frac{N_0}{2}} \cos \gamma_k \qquad \text{and} \qquad y_k = v_k \sqrt{\frac{N_0}{2}} \sin \gamma_k,$$

then the probability density function of v_k is (see Arthurs and Dym [2], p. 356)

$$p(v_1 | \theta) = v_1 \, e^{-\frac{1}{2}\left[v_1^2 + \frac{2E}{N_0} (1 + 2 c \rho \cos \theta + c^2 \rho^2)\right]} \times I_0\left(v_1 \sqrt{\frac{2E}{N_0} (1 + 2 c \rho \cos \theta + c^2 \rho^2)}\right) , \tag{38}$$

and

$$p(v_2) = v_2 \, e^{-\frac{1}{2}\left[v_2^2 + \rho^2 \frac{2E}{N_0} (1 - c)^2\right]} I_0\left(v_2 \rho \sqrt{\frac{2E}{N_0}} [1 - c]\right) , \tag{39}$$

where θ is uniformly distributed. The joint density function of v_1 and v_2, given hypothesis H_1, that frequency ω_1 was transmitted, is

$$p(v_1,\ v_2|c,\rho,H_1) =$$

$$v_1\ e^{-\frac{1}{2}\left[v_1^2 + \frac{2E}{N_0}(1 + c^2\rho^2)\right]}\ \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-\frac{2E}{N_0}\rho\ c\ \cos\theta}\ I_0\left(v_1\sqrt{\frac{2E}{N_0}(1 + 2\ c\ \rho\ \cos\theta + c^2\rho^2)}\right) d\theta$$

$$\times\ v_2\ e^{-\frac{1}{2}\left[v_2^2 + \rho^2\frac{2E}{N_0}(1 - c)^2\right]}\ I_0\left(v_2\rho\sqrt{\frac{2E}{N_0}}\,[1 - c]\right)\ . \tag{40}$$

The joint density function of v_1 and v_2, given H_2, is identical to (40) except that v_1 and v_2 are interchanged and c is replaced by 1 - c.

The optimum decision criterion is determined from the likelihood ratio, $\Lambda(\rho)$, which, for the case where the two values of c are equally probably, is

$$\Lambda(\rho) = \frac{p(v_1,v_2|0,\rho,H_2) + p(v_1,v_2|1,\rho,H_2)}{p(v_1,v_2|0,\rho,H_1) + p(v_1,v_2|1,\rho,H_1)}\ . \tag{41}$$

We define F(a,b) by

$$F(a,b) = I_0\left(a\sqrt{\frac{2E}{N_0}}\right) I_0\left(b\rho\sqrt{\frac{2E}{N_0}}\right)$$

$$\tag{42}$$

$$+\ e^{-\rho^2\frac{E}{N_0}}\ \frac{1}{2\pi}\int_{-\pi}^{\pi} I_0\left(a\sqrt{\frac{2E}{N_0}(1 + 2\ \rho\ \cos\theta + \rho^2)}\right) e^{-\rho\frac{2E}{N_0}\cos\theta}\ d\theta,$$

then, $\Lambda(\rho)$ can be expressed in terms of $F(v_1, v_2)$ by

$$\Lambda(\rho) = \frac{F\ (v_2, v_1)}{F\ (v_1, v_2)}\ . \tag{42}$$

Clearly for $v_2 = v_1$, we have

$$\Lambda(\rho) = 1\ . \tag{43}$$

Since $I_0(x)$ is monotonic increasing in x, then with $\rho \leq 1$, we have for $v_2 > v_1$ that $\Lambda\ (\rho) > 1$ and for $v_2 < v_1$ that $\Lambda\ (\rho) < 1$ so that

$v_2 > v_1$ choose hypothesis H_2

$v_2 < v_1$ choose hypothesis H_1

is the optimum receiver for equally likely hypothesis.

The P_e is calculated from

$$P_e = \int_0^\infty p\ (v_2 > v_1 \mid H_1)\ dv_1 \quad . \tag{44}$$

Using (40), we have two cases

Case (i), c = 1

$$P_{e,i} =$$

$$\int_0^\infty \frac{1}{2\pi} \int_{-\pi}^{\pi} v_1 \, e^{-\frac{1}{2}\left[v_1^2 + \frac{2E}{N_0}(1 + 2\rho\cos\theta + \rho^2)\right]} I_0\left(v_1 \sqrt{\frac{2E}{N_0}(1 + 2\rho\cos\theta + \rho^2)}\right) d\theta$$

$$\times \int_{v_1}^{\infty} v_2 \, e^{-\frac{v_2^2}{2}} \, d\,v_2 \, d\,v_1$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_0^\infty v_1 \, e^{-\frac{1}{2}\left[v_1^2 + \frac{2E}{N_0}(1 + 2\rho\cos\theta + \rho^2)\right]} I_0\left(v_1 \sqrt{\frac{2E}{N_0}(1 + 2\rho\cos\theta + \rho^2)}\right) e^{-\frac{v_1^2}{2}} dv_1 d\theta$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{2} \int_0^\infty u \, e^{-\frac{1}{2}\left[u^2 + \frac{2E}{N_0}(1 + 2\rho\cos\theta + \rho^2)\right]} I_0\left(u \sqrt{\frac{E}{N_0}(1 + 2\rho\cos\theta + \rho^2)}\right) du \, d\theta$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{2} e^{-\frac{E}{2N_0}(1 + 2\rho\cos\theta + \rho^2)} \, d\,\theta$$

$$= \frac{1}{2} e^{-\frac{E}{2N_0}(1 + \rho^2)} I_0\left(\rho \frac{E}{N_0}\right) \qquad (45)$$

Case (ii), $c = 0$

$$P_{e,ii} =$$

$$\int_0^\infty v_1\, e^{-\frac{1}{2}\left[v_1^2 + \frac{2E}{N_0}\right]} I_0\left(v_1\sqrt{\frac{2E}{N_0}}\right) \int_{v_1}^\infty v_2\, e^{-\frac{1}{2}\left(v_2^2 + \rho^2 \frac{2E}{N_0}\right)} I_0\left(v_2\ \rho\sqrt{\frac{2E}{N_0}}\right) dv_2\, dv_1$$

$$= \int_0^\infty v_1\, e^{-\left(\frac{1}{2} v_1^2 + \frac{2E}{N_0}\right)} I_0\left(v_1\sqrt{\frac{2E}{N_0}}\right) Q\left(\rho\sqrt{\frac{2E}{N_0}},\ v_1\right) dv_1 \ . \tag{46}$$

From Appendix A of Schwartz, Bennett, and Stein, [4] we have the result that (46) is equivalent to

$$P_{e,ii} = \frac{1}{2}\left[\ 1 - Q\left(\sqrt{\frac{E}{N_0}},\ \rho\sqrt{\frac{E}{N_0}}\right) + Q\left(\rho\sqrt{\frac{E}{N_0}},\ \sqrt{\frac{E}{N_0}}\right)\right] . \tag{47}$$

The P_e for DABS-like interference is therefore,

$$P_e = \frac{1}{4}\left[1 - Q\left(\sqrt{\frac{E}{N_0}},\ \rho\sqrt{\frac{E}{N_0}}\right) + Q\left(\rho\sqrt{\frac{E}{N_0}},\ \sqrt{\frac{E}{N_0}}\right) + e^{-\frac{E}{2N_0}(1 + \rho^2)} I_0\left(\rho\frac{E}{N_0}\right)\right] . \tag{48}$$

As mentioned above, ATCRBS interference is different from the DABS-like interference mainly because it occurs at frequencies other than ω_1 and ω_2. In this case, the received waveform is of the form, assuming frequency ω_1 is transmitted,

$$r(t) = \sqrt{\frac{2E}{T}} \cos(\omega_1 t + \alpha) + \rho\sqrt{\frac{2E}{T}} \cos(\omega t + \beta) + n(t) \ , \tag{49}$$

where ω is the frequency of the interfering signal. Demodulating by means of (6a) and (6b), we obtain

$$x_1 = \sqrt{E} \cos \alpha + \rho\sqrt{E}(c_1 \cos \beta - d_1 \sin \beta) + n_{x1} , \tag{50a}$$

$$y_1 = -\sqrt{E} \sin \alpha - \rho\sqrt{E}(c_1 \sin \beta + d_1 \cos \beta) + n_{y1} , \tag{50b}$$

and

$$x_2 = \rho\sqrt{E} (c_2 \cos \beta - d_2 \sin \beta) + n_{x2}, \tag{51a}$$

$$y_2 = -\rho\sqrt{E} (c_2 \sin \beta + d_2 \cos \beta) + n_{y2}, \tag{51b}$$

where

$$c_k = \frac{\sin(\omega - \omega_k) T}{(\omega - \omega_k) T} , \tag{52a}$$

$$d_k = \frac{1 - \cos(\omega - \omega_k) T}{(\omega - \omega_k) T} , \tag{52b}$$

and n_{xk}, n_{yk} are independent zero mean gaussian random variables with variance $N_0/2$.

The joint probability of x_1 and y_1 given β is

$$p(x_1,y_1|\beta) = \frac{1}{\pi N_0} e^{-\frac{1}{N}\left[x_1^2 + y_1^2 - 2\rho\sqrt{E}(x_1 a - y_1 b) + E(1 + \rho^2 \varepsilon_1)\right]}$$

$$\times\ I_0\left(\frac{\sqrt{x_1^2 + y_1^2 - 2\rho\sqrt{E}(x_1 a - y_1 b) + \rho^2 E \varepsilon_1} \quad \sqrt{E}}{\sqrt{N_0/2}}\right) , \tag{53}$$

where

$$a = c_1 \cos \beta - d_1 \sin \beta , \tag{54a}$$

$$b = c_1 \sin \beta + d_1 \cos \beta , \tag{54b}$$

and

$$\varepsilon_k = c_k^2 + d_k^2 = \frac{2[1 - \cos(\omega - \omega_k) T]}{(\omega - \omega_k)^2 T^2} . \tag{55}$$

These equations are very cumbersome, so we shall simplify by assuming the term

$$2\rho\sqrt{E}\,(x_1 a - y_1 b) ,$$

is negligible relative to the term

$$x_1^2 + y_1^2 + \rho^2 E \varepsilon_1 ,$$

and drop the former in the I_0 term. We therefore, obtain

$$p(x_1, y_1 | \beta) \approx \frac{1}{\pi N_0} e^{-\frac{1}{N_0}\left\{x_1^2 + y_1^2 - 2\rho\sqrt{E}(x_1 a - y_1 b) + E[1 + \rho^2 \varepsilon_1]\right\}}$$

$$\times I_0\left(\frac{\sqrt{x_1^2 + y_1^2 + \rho^2 E \varepsilon_1} \quad \sqrt{E}}{\sqrt{N_0/2}}\right) . \tag{56}$$

Since β is uniformly distributed, we further obtain

$$p(x_1, y_1) \approx \frac{1}{\pi N_0} e^{-\frac{1}{N_0}\left[x_1^2 + y_1^2 + E(1 + \rho^2 \varepsilon_1)\right]}$$

$$\times I_0\left(\sqrt{x_1^2 + y_1^2} \;\; \rho\sqrt{\frac{\varepsilon_1 2E}{N_0}}\right) I_0\left(\sqrt{x_1^2 + y_1^2 + \rho^2 E \varepsilon_1} \sqrt{\frac{2E}{N_0}}\right) . \tag{57}$$

Now, we let

$$x_k = v_k\sqrt{\frac{N_0}{2}} \cos \gamma_k \text{ and } y_k = v_k\sqrt{\frac{N_0}{2}} \sin \gamma_k,$$

substitute for x_1 and y_1 in (57), and integrate out γ_1

$$p(v_1) \approx v_1 \, e^{-\frac{1}{2}\left[N_1^2 + \frac{2E}{N_0}(1 + \rho^2 \varepsilon_1)\right]} I_0\left(v_1 \rho\sqrt{\frac{2E}{N_0}} \varepsilon_1\right) I_0\left(\sqrt{N_1^2 + \rho^2 E \varepsilon_1} \sqrt{\frac{2E}{N_0}}\right) . \tag{58}$$

Similarly, the density of v_2 is derived

$$p(v_2) = v_2\, e^{-\frac{1}{2}\left(v_2^2 + \frac{2E}{N_0}\rho^2\varepsilon_2\right)} I_0\left(v_2\rho\sqrt{\frac{2E}{N_0}\varepsilon_2}\right) \quad . \tag{59}$$

The probability of error, $p_{e,iii}$, is

$$P_{e,iii} = \int_0^\infty \int_{v_1}^\infty p(v_2 > v_1)\, dv_2\, dv_1$$

$$= \int_0^\infty \int_{v_1}^\infty p(v_2)\, p(v_1)\, dv_2\, dv_1 \tag{60}$$

$$\approx \int_0^\infty \nu\, e^{-\frac{1}{2}\left(\nu^2 + \frac{2E}{N_0}\{1 + \rho^2\varepsilon_1\}\right)} I_0\left(\nu\rho\sqrt{\frac{2E}{N_0}\varepsilon_1}\right) I_0\left(\sqrt{\nu^2 + \rho E\varepsilon_1}\sqrt{\frac{2E}{N_0}}\right)$$

$$\times\, Q\left(\rho\sqrt{\frac{2E}{N_0}\varepsilon_2}, \nu\right) d\nu \quad .$$

5. P_e/bit vs E/N_0

In the case of DABS-like interference, P_e/bit has been calculated for a range of E/N_0 for each of the modulation schemes. Two values for the PAM parameter, v_0, in (21) have been used. One of the values is simple

$$v_o = \sqrt{\frac{E}{2N_0}} \quad ,$$

and the other is the optimum v_0 obtained by solving (10) with $P_{20}=P_{21}=\frac{1}{2}$. The results are presented in Figure 2. Two values of ρ are used, namely $\rho = 0$ and $\rho=0.2$. It is seen that PAM is affected to a significantly greater degree than the other two modulation techniques. Using the optimum value for v_0 does not significantly improve the results obtained using

$$v_0 = \sqrt{\frac{E}{2N_0}} .$$

Figures 3 and 4 show curves of P_e/bit vs ρ with $E/N_0 = 16$ dB and 24.77 dB, respectively, for all three modulation schemes. In Figure 5 is plotted E/N_0 vs ρ necessary to achieve a P_e of 10^{-3}. We note that there is a gap between the PAM curve and the other two curves. This gap persists for different values of E/N_0. If the probability of ρ being in the range of the gap is significant, then there is a distinct advantage of DPSK or FSK to PAM.

6. CONCLUSIONS

In the above results, the maximum-likelihood estimation demodulator [1] and optimum decision criterion in an interference environment has been used. A comparison of the modulation techniques shows the optimum PAM scheme to be significantly more vulnerable to interference than the other two schemes. However, PAM cannot be summarily dismissed because of the many other aspects of the DABS link characteristics. The fixed threshold PAM-demodulators usually referred to with reference to low cost, exhibit approximately 8 dB poorer

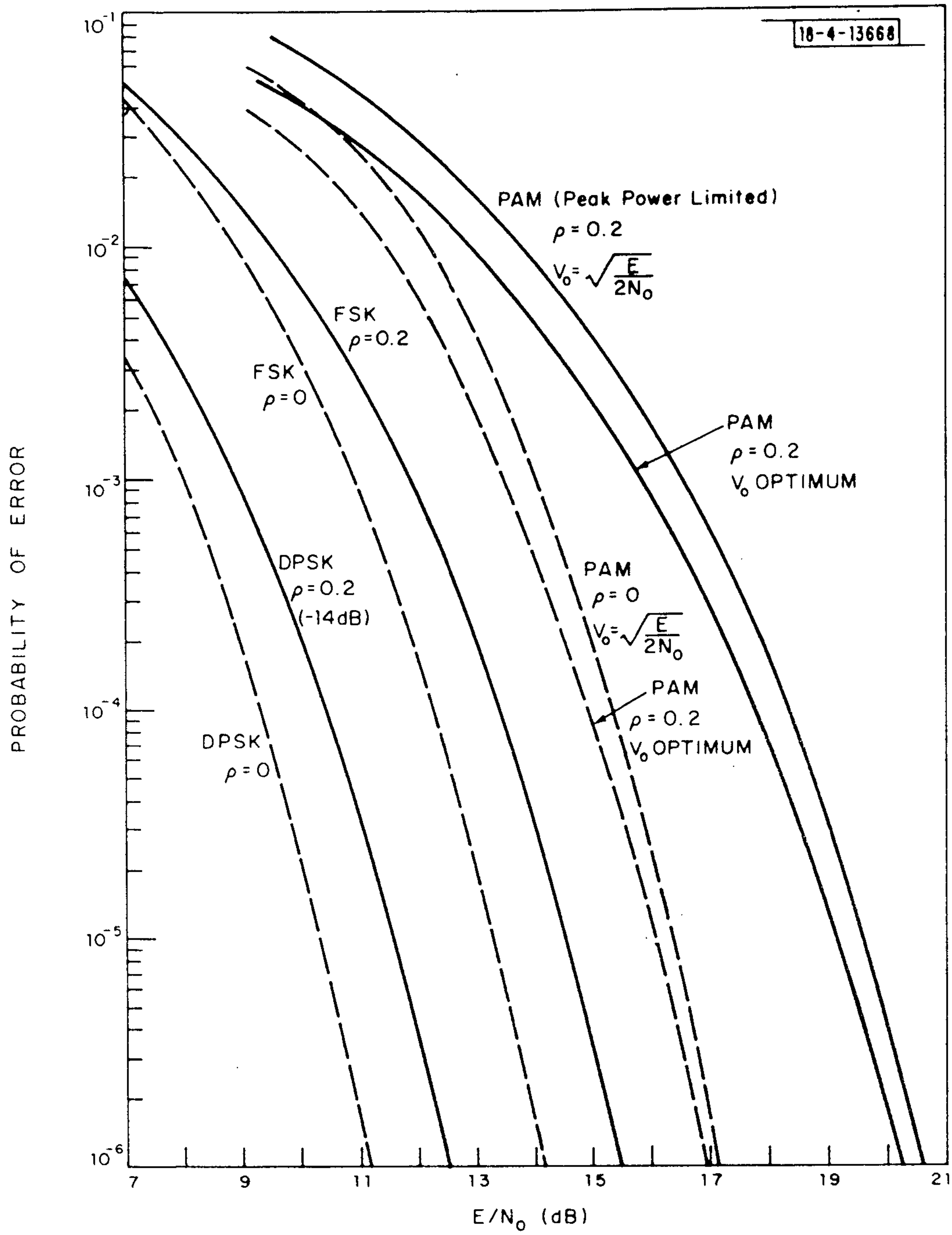


Fig. 2. P_e vs E/N_0 for $\rho = 0.0$ and $\rho = 0.2$.

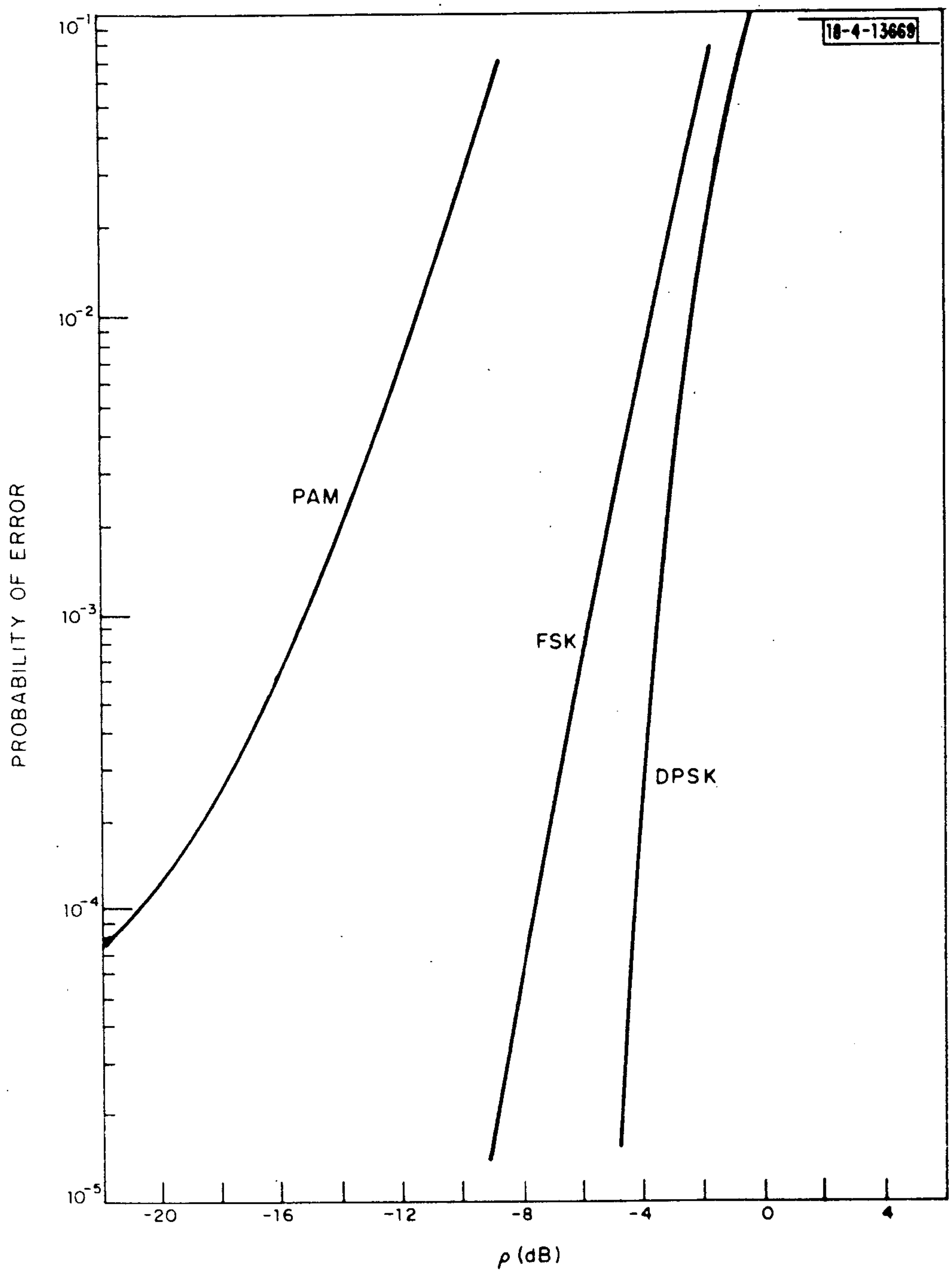


Fig. 3. P_e vs ρ for $E/N_0 = 16.02$ dB.

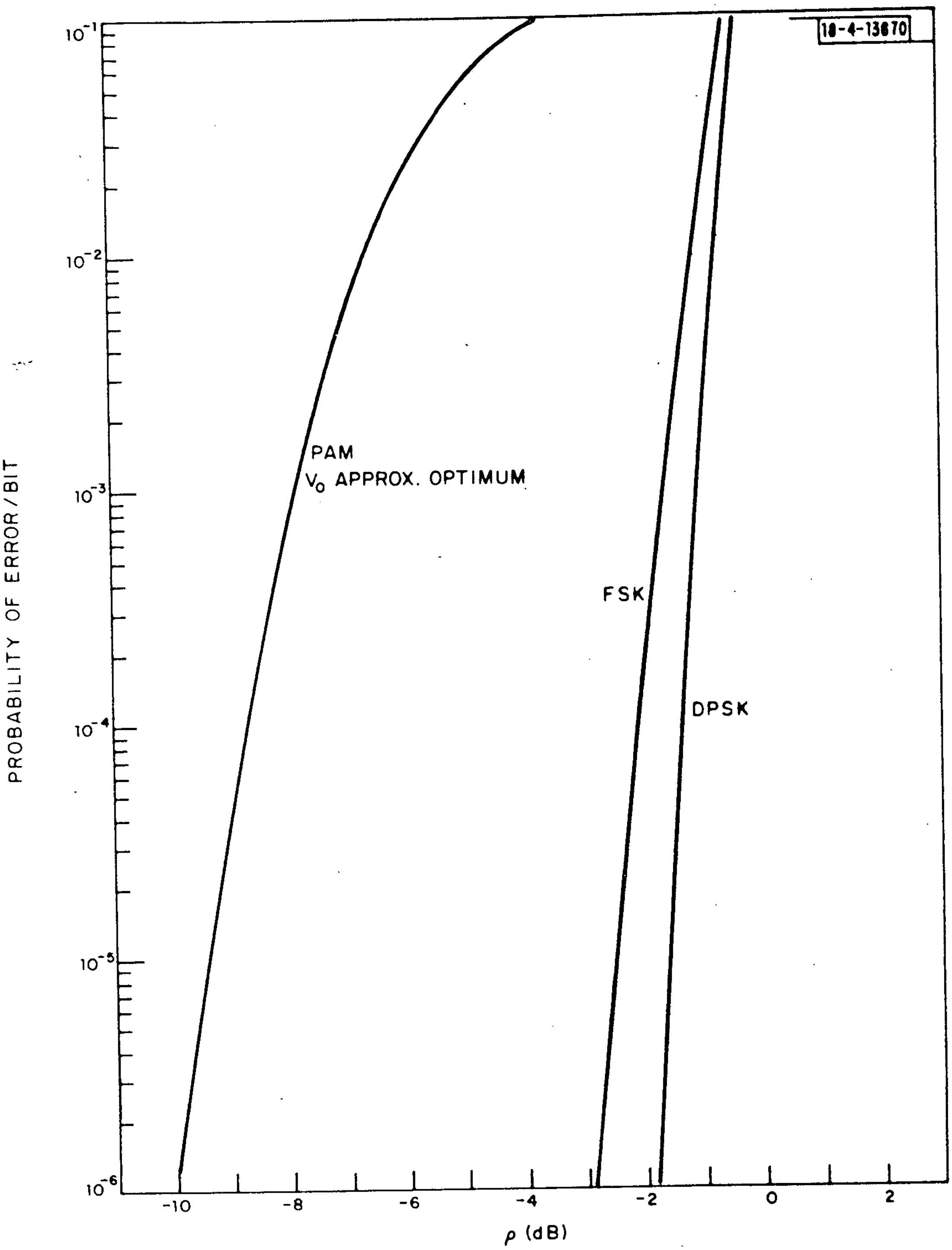


Fig. 4. P_e vs ρ for $E/N_0 = 24.77$ dB.

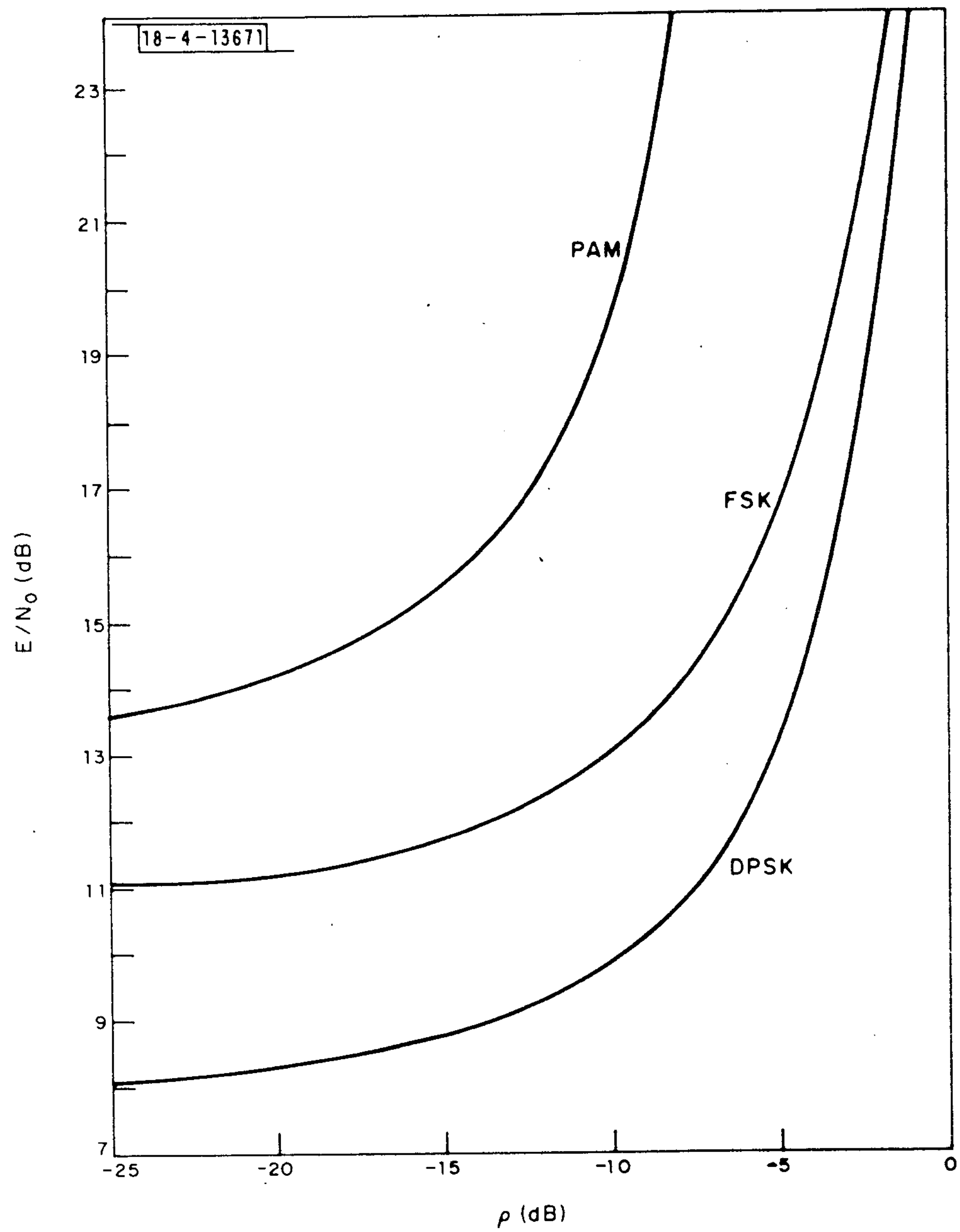


Fig. 5. Signal-to-noise ratio, E/N_0, necessary to maintain $P_e = 10^{-3}$ vs ρ.

signal-to-noise ratio performance than the optimum receiver analyzed here. FSK utilizes more bandwidth per bit than either PAM or DPSK, and transmitters are not easily implemented.

DPSK clearly has a performance advantage over both PAM and FSK from a theoretical point of view but it remains to be seen how economically it can be implemented.

The P_e/bit expressions developed here can now be used to determine P_e/message-block in ATCRBS or DABS interference. Assumptions must be made as to bit rate, message length, and interference model. For each set of assumed conditions, a P_e/block can be determined.

APPENDIX A

THE OPTIMUM DPSK RECEIVER

The details of determining the optimum DPSK receiver are presented here. We begin with the joint density of v_i and ϕ_i, given by A & D, Eq. 74, p. 351

$$p(\phi_i, v_i \,|\, \rho, \theta) = \frac{v_i}{2\pi} e^{-\frac{1}{2}\{v_i^2 - 2v_i R_+(\theta) \cos\phi_i + R_+^2(\theta)\}} .$$

We have two equally probable situations for multipath reflection interference. In the first, the phase relationship between the two reflected pulses is the same as the relationship between the two information pulses and in the second, they differ by π radians. Therefore we have

$$p(\phi_1, \phi_2, v_1, v_2 \,|\, \rho, \theta, H_0) =$$

$$\frac{1}{2}\left[\frac{v_1}{2\pi} e^{-\frac{1}{2}\{v_1^2 - 2v_1 R_+(\theta) \cos\phi_1 + R_+^2(\theta)\}} \frac{v_2}{2\pi} e^{-\frac{1}{2}\{v_2^2 - 2v_2 R_+(\theta) \cos\phi_2 + R_+^2(\theta)\}}\right.$$

$$\left. + \frac{v_1}{2\pi} e^{-\frac{1}{2}\{v_1^2 - 2\,v_1 R_+(\theta) \cos\phi_1 + R_+^2(\theta)\}} \frac{v_2}{2\pi} e^{-\frac{1}{2}\{v_2^2 - 2\,v_2 R_-(\theta) \cos\phi_2 + R_-^2(\theta)\}}\right].$$

Defining Δ as

$$\Delta = \phi_2 - \phi_1$$

we obtain

$$p(\Delta,\phi_1,v_1,v_2 \mid \rho,\theta,H_0)$$

$$= \frac{1}{2}\frac{v_1}{2\pi} e^{-\frac{1}{2}\{v_1^2 - 2 v_1 R_+(\theta) \cos \phi_1 + R_+^2(\theta)\}}$$

$$\times \left[\frac{v_2}{2\pi} e^{-\frac{1}{2}\{v_2^2 - 2v_2R_+(\theta)\cos(\Delta+\phi_1)+R_+^2(\theta)\}} + \frac{v_2}{2\pi} e^{-\frac{1}{2}\{v_2^2-2v_2R_-(\theta)\cos(\Delta+\phi_1)+R_-^2(\theta)\}} \right]$$

and similarly we obtain

$$p(\Delta,\phi_1,v_1,v_2 \mid \rho,\theta,H_1)$$

$$= \frac{1}{2}\frac{v_1}{2\pi} e^{-\frac{1}{2}\{v_1^2 - 2 v_1 R_+(\theta) \cos \phi_1 + R_+^2(\theta)\}}$$

$$\times \left[\frac{v_2}{2\pi} e^{-\frac{1}{2}\{v_2^2+2v_2R_+(\theta)\cos(\Delta+\phi_1)+R_+^2(\theta)\}} + \frac{v_2}{2\pi} e^{-\frac{1}{2}\{v_2^2+2v_2R_-(\theta)\cos(\Delta+\phi_1)+R_-^2(\theta)\}} \right]$$

The likelihood ratio $\Lambda\ (\Delta, v_1, v_2, \rho)$ is

$\Lambda(\Delta, v_1, v_2, \rho)$

$$= \frac{\displaystyle\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{-\frac{1}{2}\{R_+^2(\theta)-2v_1R_+(\theta)\cos\phi_1\}}\left[e^{-\frac{1}{2}\{R_+^2(\theta)+2v_2R_+(\theta)(\cos\Delta\cos\phi_1-\sin\Delta\sin\phi_1)\}} + e^{-\frac{1}{2}\{R_-^2(\theta)+2v_2R_-(\theta)(\cos\Delta\cos\phi_1-\sin\Delta\sin\phi_1)\}}\right] d\phi_1 d\theta}{\displaystyle\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{-\frac{1}{2}\{R_+^2(\theta)-2v_1R_+(\theta)\cos\phi_1\}}\left[e^{-\frac{1}{2}\{R_+^2(\theta)+2v_2R_+(\theta)[\cos(\Delta+\pi)\cos\phi_1-\sin\Delta\sin\phi_1]\}} + e^{-\frac{1}{2}\{R_-^2(\theta)+2v_2R_-(\theta)[\cos(\Delta+\pi)\cos\phi_1-\sin\Delta\sin\phi_1]\}}\right] d\phi_1 d\theta}$$

where in the denominator we have used the following:

$$\sin(\Delta+\pi)\sin\phi = \sin\Delta\sin(-\phi)$$

and replaced $-\phi$ by ϕ_1.

We see that when $-\pi/2 < \Delta < \pi/2$ then $\cos\Delta$ is greater than zero and $\cos(\Delta+\pi)$ less than zero so $\Lambda < 1$ while for $\Delta > \pi/2$ or $\Delta < -\pi/2$, $\cos\Delta < 1$ and $\cos(\Delta+\pi) > 1$ so that $\Lambda > 1$ independent of v_1, v_2, R_+ and R_-.

In the case of ATCRBS interference where only a single pulse is interfered with, we have

$$p(\Delta,\phi_1,v_1,v_2\,|\,\rho,\theta,H_0)$$

$$= \frac{v_1}{2\pi} e^{-\frac{1}{2}\{v_1^2-2v_1R_+(\theta)+R_+^2(\theta)\}} \frac{v_2}{2\pi} e^{-\frac{1}{2}\{v_2^2-2v_2\sqrt{\frac{E}{N_0}}\cos(\Delta+\phi_1)+\frac{E}{N_0}\}}$$

and becomes

$$\Lambda(\Delta,v_2) =$$

$$\frac{\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{-\frac{1}{2}\{R_+^2(\theta)-2v_1R_+(\theta)\cos\phi\}} d\theta \left[e^{-\frac{1}{2}\{\frac{E}{N_0}+2v_2\sqrt{\frac{E}{N_0}}(\cos\Delta\,\cos\phi_1-\sin\Delta\,\sin\phi_1)\}}\right] d\phi_1}{\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{-\frac{1}{2}\{R_+^2(\theta)-2v_1R_+(\theta)\cos\phi_1\}} d\theta \left[e^{-\frac{1}{2}\{\frac{E}{N_0}+2v_2\sqrt{\frac{E}{N_0}}(\cos(\Delta+\pi)\cos\phi_1-\sin\Delta\,\sin\phi_1)\}}\right] d\phi_1}$$

The receiver is the same as before.

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16. Abstract

The performance of three candidate modulation schemes for DABS is analyzed in this report and a comparison on the basis of probability of error per bit is made. The three types of modulation are PAM, DPSK, and FSK. The results show that, at a given level of interference and signal-to-noise ratio, DPSK and FSK have a lower P_e/bit than PAM and this difference is significant in most cases. In addition to P_e/bit, however, the choice of modulation and message format depends on the capacity required, bandwidth occupancy, and cost of implementation. This list consideration is especially important with regard to the transponder.

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