

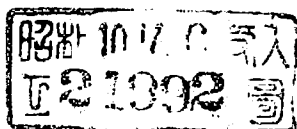
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THICKNESS VIBRATION OF PIEZOELECTRIC  
OSCILLATING CRYSTAL.

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## THICKNESS VIBRATION OF PIEZOELECTRIC OSCILLATING CRYSTAL.

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### 1. Introduction.

For several years since, it is well known that in the natural vibration of piezoelectric oscillating crystal, there is a certain period proportional only to the thickness, but, as far as we are aware, no concrete explanation for the mode of vibration corresponding to this period and how this period is connected to the physical constants of the medium. The present paper shows that such a kind of vibration, named "Thickness Vibration" for brevity, is due to the standing wave produced by interference of plane waves incident to and reflected from the plane boundary surfaces of the medium, and verifies the theoretical results by several examples.

### 2. Equations of Motions in Crystalline Media.

We must first emphasize that to solve the thickness vibration of a very thin crystal plate it is very important to notice that the period of vibration is practically dependent upon but the thickness of the plate, that is, independent upon the extent of the major surfaces of the plate; this fact shows that the free vibration of such a kind is what can be realized if the major surfaces of the plate extend to infinity.

Now the free vibration is, in general, a result of an occurrence of a standing wave in the medium, and this standing wave is produced by the interference of two similar waves propagating in the opposite directions, that is, the interference of waves incident to and reflected from the boundary surfaces of the medium.

Such is the case, the free vibration of thin crystal plate having two parallel plane boundary surfaces is due to the interference of plane waves incident to and reflected from those boundary plane surfaces of the medium. In other words, in the thickness vibration of the crystal plate bounded by the two infinite parallel planes the displacement at any point in the medium is dependent only upon the distance from a boundary surface and the



If  $\lambda$ ,  $\mu$ ,  $\nu$  and  $c$  be so chosen that the relations

$$\left. \begin{aligned} \lambda P^2 + \mu PQ + \nu PR &= \lambda c, \\ \lambda PQ + \mu Q^2 + \nu QR &= \mu c, \\ \lambda PR + \mu QR + \nu R^2 &= \nu c, \end{aligned} \right\} \quad (7)$$

hold, then multiplying the three equations of (4) by  $\lambda$ ,  $\mu$ ,  $\nu$ , and adding together, we have in the most compact form:

$$\frac{\partial^2 \xi}{\partial t^2} = \frac{c}{\rho} \frac{\partial^2 \xi}{\partial s^2} \quad (8)$$

Eliminating  $\lambda$ ,  $\mu$ ,  $\nu$  between the three equations of (7), we have

$$\begin{vmatrix} P^2 - c & PQ & PR \\ PQ & Q^2 - c & QR \\ PR & QR & R^2 - c \end{vmatrix} = 0, \quad (9)$$

a cubic equation determining  $c$ ; let its roots, which are proved to be necessarily real and positive,<sup>1)</sup> be  $c_1$ ,  $c_2$ ,  $c_3$ . When  $c_1$  is substituted in any two of the foregoing equations, the ratios of  $\lambda : \mu : \nu$  can be derived; let them be denoted by  $\lambda_1 : \mu_1 : \nu_1$ , and suppose the corresponding value of  $\xi$  to be  $\xi_1$ ; with similar expressions for the other values of  $c$ . Then  $\xi_1$ ,  $\xi_2$  and  $\xi_3$  are always proved to be perpendicular to each others.

The solution of (8) is a very simple matter. To determine the normal mode of vibration we must assume that  $\xi$  varies as a harmonic function of the time  $t$

$$\xi \propto e^{ipt}, \quad (10)$$

then as a function of  $s$ ,  $\xi$  must satisfy

$$\frac{d^2 \xi}{ds^2} + \frac{p^2 \rho}{c} \xi = 0, \quad (11)$$

of which the complete integral is

$$\xi = A \cos \left( sp \sqrt{\frac{\rho}{c}} \right) + B \sin \left( sp \sqrt{\frac{\rho}{c}} \right), \quad (12)$$

where  $A$  and  $B$  are independent of  $s$ .

Now when both boundary surfaces are free from tractions,

$$\frac{d\xi}{ds} = 0 \quad \text{at } s=0 \quad \text{and } s=a, \quad (13)$$

$$\text{we get} \quad B=0 \quad \text{and} \quad \sin \left( ap \sqrt{\frac{\rho}{c}} \right) = 0, \quad (14)$$

1) Lord KELVIN: Baltimore Lectures, London 1904.

from which 
$$\frac{p}{2\pi} = \frac{q}{2a} \sqrt{\frac{c}{\rho}}, \quad (15)$$

$q$  being intergral.

Accordingly, the normal mode of vibration is given by equation of a form

$$\xi = A \cos\left(q\pi \frac{z}{a}\right) \cdot e^{ipt} \quad (16)$$

where  $A$  is an arbitrary constant, which may be determined in the usual manner, when the initial value of  $\xi$  is given.

It is well known that we can excite one or more of the free thickness vibrations by applying an alternating electric field of suitable frequency in the direction normal to the boundary surfaces of crystal, provided the crystal plate is prepared so that it is piezoelectrically active; and also that it is sometimes possible to sustain a continuous oscillation by properly employing the thermionic tube circuit, the easeness of maintaining being considerably dependent upon the medium and the orientation of the boundary surfaces.

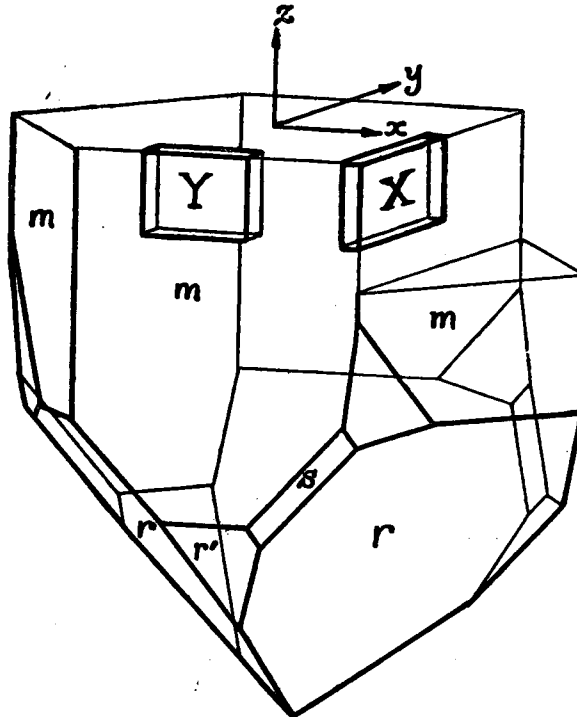


Fig. 1.

### 3. Thickness Vibration of Quartz Plate.

Of the piezoelectric crystals, quartz is the most familiar to high frequency engineers, so that we may first verify the theory by this medium. In the quartz, if the co-ordinate axes be chosen as in Fig. 1, the adiabatic elastic constants<sup>2)</sup> and density are as follows:

$$\begin{array}{cccccc}
 c_{11} & c_{12} & c_{13} & c_{14} & 0 & 0 \\
 & c_{11} & c_{13} & -c_{14} & 0 & 0 \\
 & & c_{33} & 0 & 0 & 0 \\
 & & & c_{44} & 0 & 0 \\
 & & & & c_{44} & c_{14} \\
 & & & & & \frac{1}{2}(c_{11} - c_{12})
 \end{array} \tag{17}$$

$$\left. \begin{array}{l}
 c_{11} = 85.45 \times 10^{10} \text{ dynes/cm}^2, \quad c_{12} = 7.26 \times 10^{10} \text{ dynes/cm}^2, \\
 c_{33} = 105.67 \times 10^{10} \text{ dynes/cm}^2, \quad c_{13} = 14.37 \times 10^{10} \text{ dynes/cm}^2, \\
 c_{44} = 57.09 \times 10^{10} \text{ dynes/cm}^2, \quad -c_{14} = 16.87 \times 10^{10} \text{ dynes/cm}^2, \\
 \rho = 2.654 \text{ grams/cm}^2,
 \end{array} \right\} \tag{18}$$

therefore in the general expression (9) to determine the values of  $c$

$$\left. \begin{array}{l}
 P^2 = l^2 c_{11} + \frac{1}{2} m^2 (c_{11} - c_{12}) + n^2 c_{44} + 2mnc_{14}, \\
 Q^2 = \frac{1}{2} l^2 (c_{11} - c_{12}) + m^2 c_{11} + n^2 c_{44} - 2mnc_{14}, \\
 R^2 = (l^2 + m^2) c_{44} + n^2 c_{33}, \\
 PQ = 2nlc_{14} + \frac{1}{2} lm(c_{11} + c_{12}), \\
 QR = (l^2 - m^2) c_{14} + mn(c_{13} + c_{44}), \\
 PR = 2lmc_{14} + nl(c_{13} + c_{44}).
 \end{array} \right\} \tag{19}$$

By the way, the relations between applied electric fields and strains are:

$$\left. \begin{array}{l}
 e_{xx} = d_{11} E_x, \quad e_{xx} = -d_{14} E_y, \\
 e_{yy} = -d_{11} E_x, \quad e_{xy} = -2d_{11} E_y, \\
 e_{yz} = d_{14} E_x, \\
 d_{11} = 6.45 \times 10^{-8} \text{ C.G.S. units,} \\
 -d_{14} = 1.45 \times 10^{-8} \text{ C.G.S. units.}
 \end{array} \right\} \tag{20}$$

(1) If a quartz crystal be cut by two planes perpendicular to the axis  $x$  to obtain a so-called  $X$ -cut or Curie-cut quartz oscillating crystal designated by  $X$  in Fig. 1, since the direction cosines of normal to the major surfaces is seen to be chosen as:

2) W. VOIGT: Lehrbuch der Kristallphysik. 1928 s. 754 u. 789.



$$l=1, \quad m=n=0, \quad (21)$$

six quantities of (19) reduce to

$$\left. \begin{aligned} P^2 &= c_{11}, & PQ &= 0, \\ Q^2 &= \frac{1}{2}(c_{11} - c_{12}), & QR &= c_{14}, \\ R^2 &= c_{44}, & PR &= 0. \end{aligned} \right\} \quad (22)$$

Introducing these quantities into (9), we get

$$\left. \begin{aligned} c_1 &= c_{11} = 85.45 \times 10^{10} \text{ dynes/cm}^2, \\ c_2, c_3 &= \frac{1}{2}(c_{11} - c_{12}) + \frac{1}{2}c_{44} \mp \sqrt{\left\{ \frac{1}{4}(c_{11} - c_{12}) - \frac{1}{2}c_{44} \right\}^2 + c_{14}^2}, \\ c_2 &= 28.98 \times 10^{10} \text{ dynes/cm}^2, \\ c_3 &= 67.22 \times 10^{10} \text{ dynes/cm}^2, \end{aligned} \right\} \quad (23)$$

and from (7),

$$\left. \begin{aligned} \lambda_1 &= 1, & \mu_1 &= \nu_1 = 0, \\ \lambda_2 &= 0, & \mu_2 &= 0.857, & \nu_2 &= 0.515, \\ \lambda_3 &= 0, & \mu_3 &= -0.515, & \nu_3 &= 0.857. \end{aligned} \right\} \quad (24)$$

The normal modes of vibrations are given by the following equations:

$$\left. \begin{aligned} \xi &= A \cos\left(q\pi \frac{x}{a}\right) \cdot e^{ipt}, \\ \text{where } \xi_1 &= u, \\ \xi_2 &= 0.857v + 0.515w, \\ \xi_3 &= -0.515v + 0.857w, \end{aligned} \right\} \quad (25)$$

while the electric field in the direction  $x$  cannot excite but the displacement  $u$  as is seen from (20), so that the frequency of vibration is

$$\frac{p}{2\pi} = \frac{q}{2a} \sqrt{\frac{c_{11}}{\rho}} = \frac{q}{a} \times 0.284 \times 10^6 \text{ cycles/sec.}, \quad (26)$$

and the mode of vibrations is

$$u = A \cos\left(q\pi \frac{x}{a}\right) \cdot e^{ipt}, \quad (27)$$

that is, the pure longitudinal vibration along the normal to the boundary surfaces.

As an experimental example, a quartz oscillating crystal  $0.0922 \times 2.610 \times 2.684 \text{ cm}^3$  (the normal to the major surfaces inclines  $7'$  to the axis  $x$ ) has the fundamental frequency of thickness vibration  $3.10 \times 10^6 \text{ cycles/sec.}$ , from which we get

$$\frac{p}{2\pi} = \frac{1}{a} \times 0.0922 \times 3.10 \times 10^6 = \frac{1}{a} \times 0.286 \times 10^6 \text{ cycles/sec.}, \quad (28)$$

which is seen to be very near upon the theoretical result.

(2) In *Y*-cut quartz plate designated by *Y* in Fig. 1, the direction cosines of normal to the major surfaces are chosen as

$$m=1, \quad l=n=0, \quad (29)$$

hence six quantities of (19) become

$$\left. \begin{aligned} P^2 &= \frac{1}{2}(c_{11} - c_{12}), & PQ &= 0, \\ Q^2 &= c_{11}, & QR &= -c_{14}, \\ R^2 &= c_{44}, & PR &= 0. \end{aligned} \right\} \quad (30)$$

Introducing these quantities into (9), we get

$$\left. \begin{aligned} c_1 &= \frac{1}{2}(c_{11} - c_{12}) = 39.10 \times 10^{10} \text{ dynes/cm}^2, \\ c_2, c_3 &= \frac{1}{2}(c_{11} + c_{44}) \pm \sqrt{\frac{1}{4}(c_{11} - c_{44})^2 + c_{14}^2}, \\ c_2 &= 93.31 \times 10^{10} \text{ dynes/cm}^2, \\ c_3 &= 49.23 \times 10^{10} \text{ dynes/cm}^2, \end{aligned} \right\} \quad (31)$$

and from (7),

$$\left. \begin{aligned} \lambda_1 &= 1, & \mu_1 &= \nu_1 = 0, \\ \lambda_2 &= 0, & \mu_2 &= -0.907, & \nu_2 &= -0.422, \\ \lambda_3 &= 0, & \mu_3 &= 0.422, & \nu_3 &= -0.907. \end{aligned} \right\} \quad (32)$$

The normal modes of vibrations are given by

$$\text{where } \left. \begin{aligned} \xi &= A \cos\left(q\pi \frac{y}{a}\right) \cdot e^{i p t}, \\ \xi_1 &= u, \\ \xi_2 &= -0.907v - 0.422w, \\ \xi_3 &= 0.422v - 0.907w, \end{aligned} \right\} \quad (33)$$

of which  $\xi_2$  and  $\xi_3$  cannot be excited piezoelectrically by the electric field in the direction of *y* axis as is clear from (20), so that the displacement is given by

$$u = A \cos\left(q\pi \frac{y}{a}\right) \cdot e^{i p t}, \quad (34)$$

which shows that the vibration is of pure shear as is pointed out by Prof. Dr. Cady.<sup>3)</sup>

3) W. G. CADY: A shear mode of crystal vibration (abstract) Phys. Rev. Vol. 29, p. 617, 1927.

The frequency of vibration is

$$\frac{p}{2\pi} = \frac{q}{a} \sqrt{\frac{c_{11} - c_{12}}{2\rho}} = \frac{q}{a} \times 0.192 \times 10^6 \text{ cycles/sec.}, \quad (35)$$

while in the experiment by a sample  $0.1909 \times 3.200 \times 3.304 \text{ cm}^3$  (the normal to the major surfaces inclines  $6'$  to the axis  $y$ ) the fundamental frequency of thickness vibration is  $1.02 \times 10^6 \text{ cycles/sec.}$ , that is

$$\frac{p}{2\pi} = \frac{1}{a} \times 0.1909 \times 1.02 \times 10^6 = \frac{1}{a} \times 0.195 \times 10^6 \text{ cycles/sec.}, \quad (36)$$

which is also seen to be very near upon the theoretical result (35).

(3) In quartz plate cut parallel to a surface of positive rhombohedron designated by  $r$  in Fig. 1 (say  $R$ -cut plate<sup>4)</sup>), the vibration is observed to be extremely vigorous and the third harmonic vibration ( $q=3$ ) can often be sustained by Pierce circuit,<sup>5)</sup> in which an oscillating crystal plate is placed between grid and filament of a three electrode thermionic tube.

In this case the direction cosines of normal to the major surfaces are:

$$\left. \begin{aligned} l=0, \quad m = -\sin \theta, \quad n = \cos \theta, \\ \text{where} \quad \theta = \arctan \frac{2.200}{\sqrt{3}}. \end{aligned} \right\} \quad (37)$$

as shown in Fig. 2, hence six quantities of (19) become

$$\left. \begin{aligned} P^2 &= \frac{1}{2}(c_{11} - c_{12}) \sin^2 \theta + c_{44} \cos^2 \theta - c_{14} \sin 2\theta, \\ Q^2 &= c_{11} \sin^2 \theta + c_{44} \cos^2 \theta + c_{14} \sin 2\theta, \\ R^2 &= c_{44} \sin^2 \theta + c_{33} \cos^2 \theta, \\ PQ &= 0, \\ QR &= -c_{14} \sin^2 \theta - (c_{13} + c_{43}) \sin \theta \cos \theta, \\ PR &= 0. \end{aligned} \right\} \quad (38)$$

Introducing these into (9), we get

$$\left. \begin{aligned} c_1 &= \frac{1}{2}(c_{11} - c_{12}) \sin^2 \theta + c_{44} \cos^2 \theta - c_{14} \sin 2\theta = 62.39 \times 10^{10} \text{ dynes/cm}^2, \\ c_2 &= 41.10 \times 10^{10} \text{ dynes/cm}^2, \\ c_3 &= 92.78 \times 10^{10} \text{ dynes/cm}^2, \end{aligned} \right\} \quad (39)$$

and from (7),

4) I. KOGA:  $R$ -cut quartz oscillating crystal plate and the harmonic vibration.—Supplementary Issue, Jour. I.E.E. Japan, pp. 170–172, April, 1932.

5) G. W. PIERCE: Piezoelectric crystal resonators and crystal oscillators applied to the precision calibration of wave-meters—Proc. A. A. A. S. Vol. 59, pp. 81–106, 1923.

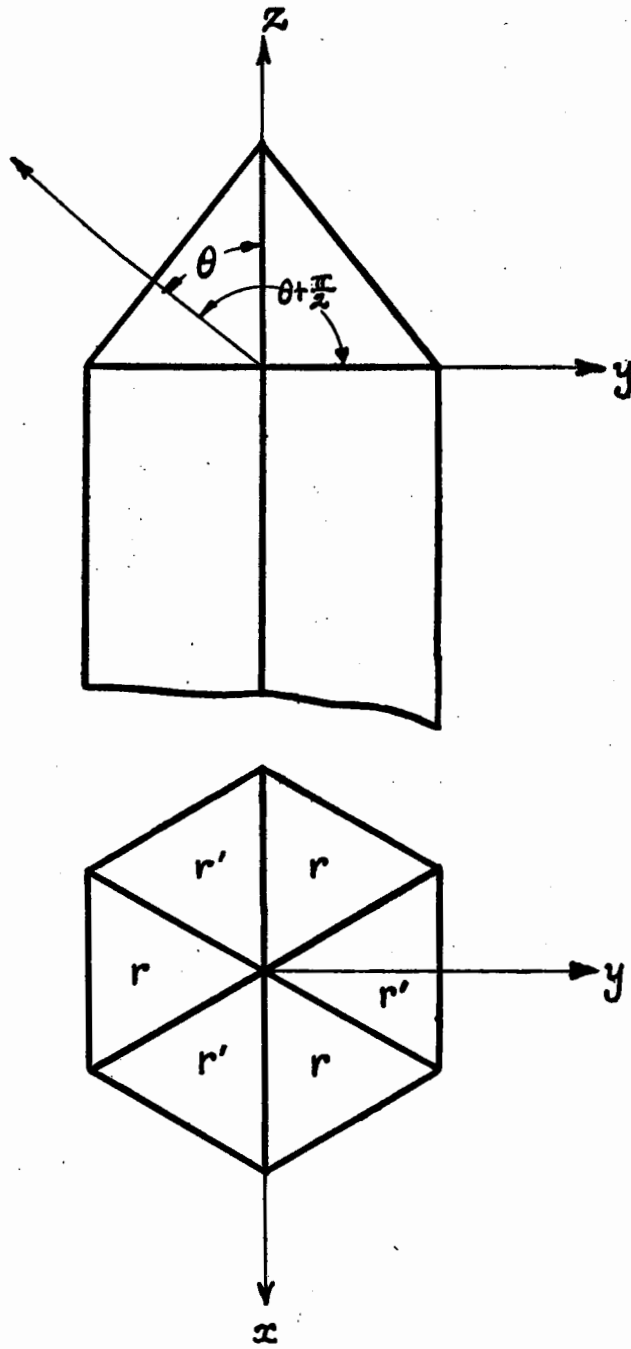


Fig. 2.

$$\left. \begin{aligned} \lambda_1=1, \quad \mu_1=\nu_1=0, \\ \lambda_2=0, \quad \mu_2=-0.818, \quad \nu_2=-0.575, \\ \lambda_3=0, \quad \mu_3=-0.575, \quad \nu_3=0.818. \end{aligned} \right\} \quad (40)$$

The normal modes of vibrations are given by

$$\left. \begin{aligned} \xi &= A \cos\left(q\pi \frac{-y \sin \theta + z \cos \theta}{a}\right) \cdot e^{ipt}, \\ \text{where } \xi_1 &= u, \\ \xi_2 &= -0.818v - 0.575w, \\ \xi_3 &= -0.575v + 0.818w, \end{aligned} \right\} \quad (41)$$

of which  $\xi_2$  and  $\xi_3$  cannot be excited piezoelectrically by the electric field along  $yz$  plane as is seen from (20), so that the displacement is given by

$$u = A \cos\left(q\pi \frac{-y \sin \theta + z \cos \theta}{a}\right) \cdot e^{ipt}, \quad (42)$$

which shows that the mode of vibration is of pure shear. The frequency of vibration is

$$\frac{p}{2\pi} = \frac{q}{2a} \sqrt{\frac{c_1}{\rho}} = \frac{q}{a} \times 0.242 \times 10^6 \text{ cycles/sec.}, \quad (43)$$

while in the experiment by a sample  $0.1002 \times 2.592 \times 2.762 \text{ cm}^3$  (the major surfaces inclines within  $1'$  to a surface of positive rhombohedron  $r$ ) the fundamental frequency of thickness vibration is  $2.47 \times 10^6$  cycles/sec., that is

$$\frac{p}{2\pi} = \frac{1}{a} \times 0.1002 \times 2.47 \times 10^6 = \frac{1}{a} \times 0.247 \times 10^6 \text{ cycles/sec.}, \quad (44)$$

which is also seen to be very near upon the theoretical result (43).

There have been published several papers<sup>6)</sup> upon the excitation of harmonic vibrations of a quartz oscillating crystal as a resonator, but as far as we are aware, none has yet succeeded to sustain harmonic vibration, and that in the thickness vibration, of crystal itself as an oscillator, while the present crystal plate can often be sustained its vibrations of the first and third harmonic separately by the Pierce circuit above-mentioned. Fig. 3 shows the characteristic features of the oscillator. The higher frequency of vibration is easily observed by heterodyne frequency meter to be just three times that of the lower.

6) E. GIEBE und A. SCHEIBE: Sichtbarmachung von hochfrequenten Longitudinalschwingungen piezoelektrischer Kristalltaeche. ZS. f. Phys. Bd. 33, s. 335, 1925. etc.

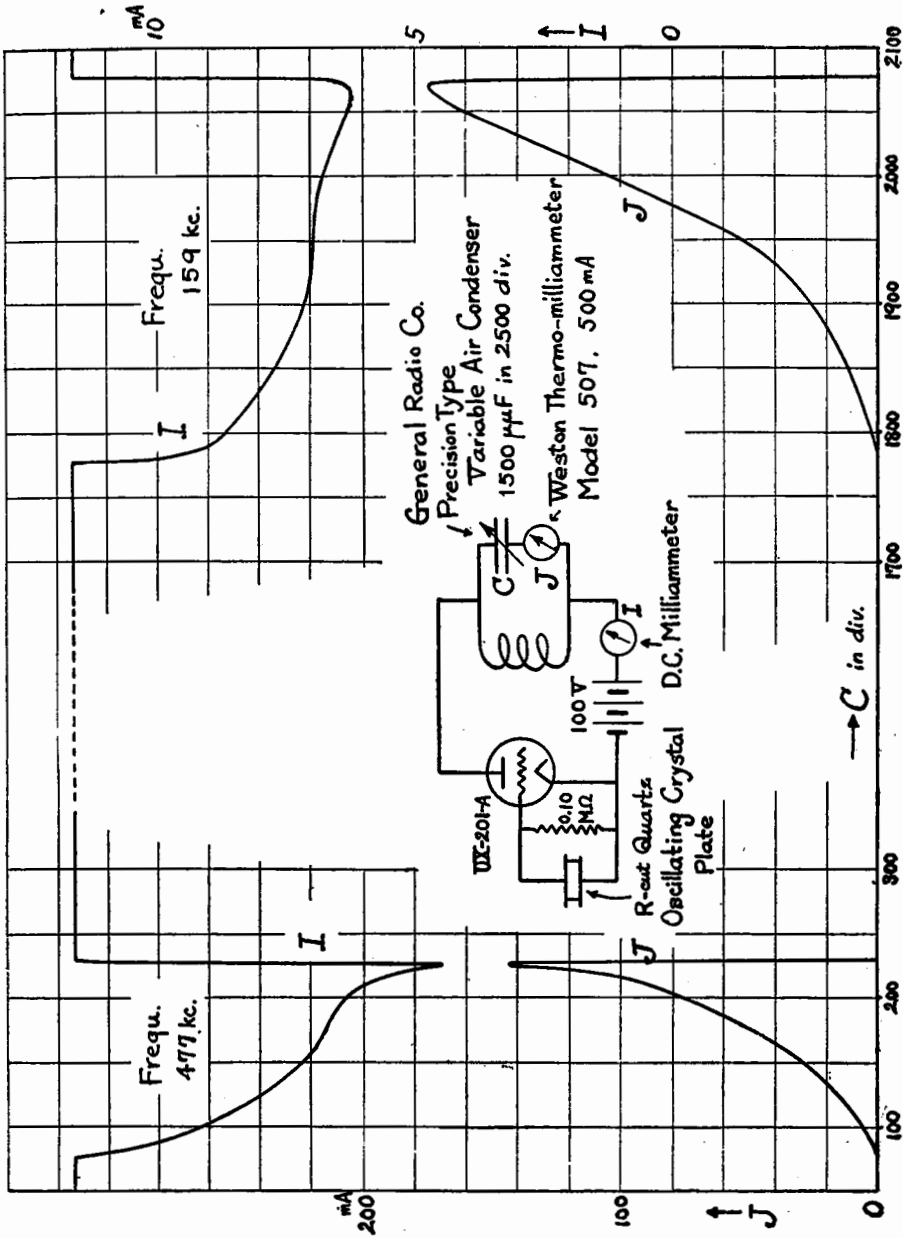


Fig. 3.

Examples of dimensional proportions<sup>7)</sup> which are suited for the third harmonic vibrations are :

Dimensions (cm <sup>3</sup> )	Approx. Frequency of Vibrations (cycles/sec.)		
0.052 × 2.26 × 2.84	4.80 × 10 <sup>6</sup>	and	4.80 × 10 <sup>6</sup> × 3
0.069 × 2.24 × 2.56	3.55 × 10 <sup>6</sup>	and	3.55 × 10 <sup>6</sup> × 3
0.093 × 2.56 × 2.57	2.66 × 10 <sup>6</sup>	and	2.66 × 10 <sup>6</sup> × 3
0.114 × 2.09 × 2.70	2.17 × 10 <sup>6</sup>	and	2.17 × 10 <sup>6</sup> × 3
0.155 × 2.47 × 2.81	1.59 × 10 <sup>6</sup>	and	1.59 × 10 <sup>6</sup> × 3
0.185 × 2.77 × 2.98	1.34 × 10 <sup>6</sup>	and	1.34 × 10 <sup>6</sup> × 3

(45)

(Inclinations of the major surfaces to a surface of positive rhombohedron  $r$  are generally not greater than 10°).

(4) If the boundary surfaces are parallel to a surface of negative rhombohedron designated by  $r'$  in Fig. 1, (say  $R'$ -cut plate) normal modes and frequencies of vibrations can be studied quite similarly as in the previous case, the only difference being the reversal of the sign of  $\theta$  in Fig. 2. Thus

$$c_1 = \frac{1}{2}(c_{11} - c_{12}) \sin^2 \theta + c_{44} \cos^2 \theta + c_{14} \sin 2\theta = 29.59 \times 10^{10} \text{ dynes/cm}^2, \quad (46)$$

and

$$\frac{p}{2\pi} = \frac{q}{2a} \sqrt{\frac{c_1}{\rho}} = \frac{q}{a} \times 0.167 \times 10^6 \text{ cycles/sec.} \quad (47)$$

The displacement

$$u = A \cos \left( q\pi \frac{y \sin \theta + z \cos \theta}{a} \right) \cdot e^{ipt} \quad (48)$$

gives a pure shear vibration.

From the experiment with a sample  $0.0765 \times 2.76 \times 3.19 \text{ cm}^3$  (the major surfaces inclines  $5'$  to a surface of negative rhombohedron  $r'$ ) the frequency of vibration is  $2.20 \times 10^6$  cycles/sec., that is

$$\frac{p}{2\pi} = \frac{1}{a} \times 0.0765 \times 2.20 \times 10^6 = \frac{1}{a} \times 0.168 \times 10^6 \text{ cycles/sec.}, \quad (49)$$

which is very near upon the theoretical results (47).

It is seen above that the frequency of vibration measured as a piezoelectric oscillating crystal is always little higher than that calculated from elastic constants. This small difference must be, to a certain extent, due to the finiteness of the boundary surfaces of plate, but it is not overlooked that the frequency of pure mechanical vibration is different from that as a

7) I. KOGA: Note on the Piezoelectric Quartz Oscillating Crystal regarded from the Principle of Similitude—Journ. I. E. E. Japan, Supplementary Issue, pp. 27-28, April, 1931; Proc. I. R. E. Vol. 19, No. 6, pp. 1022-1023, June, 1931.

piezoelectric oscillating crystal. In fact, when crystal plate is excited to vibrate with its natural vibration, the equivalent elastic constants become slightly greater than that which is determined elasto-statically. For example, by strain  $e_{xx}$ , besides the stress  $c_{11}e_{xx}$ , the polarization of  $\epsilon_{11}e_{xx}$  is produced,  $\epsilon_{11}$  being the piezoelectric constant, so that the crystal, being acted by the electric field of  $4\pi\epsilon_{11}e_{xx}/K$ , where  $K$  is the dielectric constant, gives the stress  $4\pi\epsilon_{11}^2e_{xx}/K$ , which adds to the above-mentioned stress  $c_{11}e_{xx}$ . Consequently the resultant adiabatic elastic constant  $c_{11}^0$  becomes

$$c_{11}^0 = c_{11} + \frac{4\pi\epsilon_{11}^2}{K} = c_{11} \left\{ 1 + \frac{4\pi \times (4.77 \times 10^4)^2}{85.45 \times 10^{10} \times 4.5} \right\} = c_{11}(1 + 0.0074) \quad (50)$$

In other words, to replace a quartz oscillating crystal by an equivalent electrical network,<sup>8)</sup> we must, as shown in Fig. 4, consider  $C_1$ , which represents the capacity of the quartz as a mere dielectric, parallel to the series resonance circuit  $L_0, C_0, R_0$  representing the mechanical vibrating system. Such is the case, we can easily understand that the electrically measured natural frequency of vibration determined by  $L_0, C_0, C_1$  is always higher than that determined by  $L_0, C_0$ . The ratio  $C_0/C_1$  is, in general, not greater than 1/100.

#### 4. Thickness Vibration of Tourmaline Plate.

Tourmaline is not yet widely used as piezoelectric oscillating crystal, and we have no occasion to investigate by fine crystal, but we can foresee the normal modes and frequencies of vibrations in the first approximation.

The adiabatic elastic constants referred to the co-ordinate axes shown

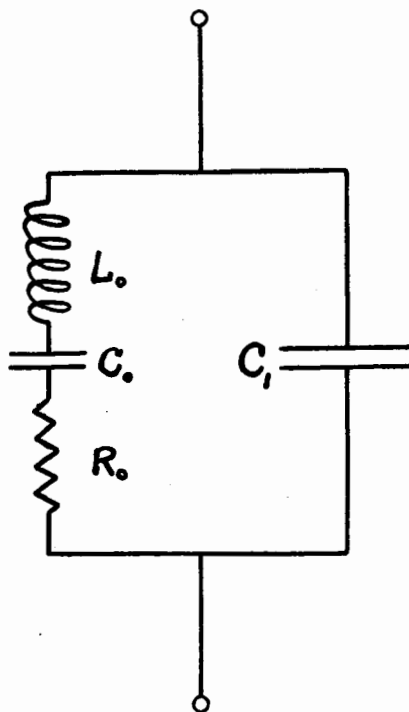


Fig. 4.

8) D. W. DYE: Piezoelectric quartz resonator and equivalent electrical circuit—Proc. Phys. Soc., Vol. 38, pp. 399-457, 1926.



in Fig. 5 are composed of the same scheme with (17), therefore the general expression to determine the values of  $c$  is the same with that of quartz. The numerical values of adiabatic elastic constants<sup>2)</sup> and density are

$$\left. \begin{aligned} c_{11} &= 270.17 \times 10^{10} \text{ dynes/cm}^2, \\ c_{33} &= 160.69 \times 10^{10} \text{ dynes/cm}^2, \\ c_{44} &= 66.71 \times 10^{10} \text{ dynes/cm}^2, \\ c_{12} &= 69.06 \times 10^{10} \text{ dynes/cm}^2, \\ c_{13} &= 8.83 \times 10^{10} \text{ dynes/cm}^2, \\ -c_{14} &= 7.75 \times 10^{10} \text{ dynes/cm}^2, \end{aligned} \right\} (51)$$

and  $\rho = 3.100 \text{ grams/cm}^3$  say.

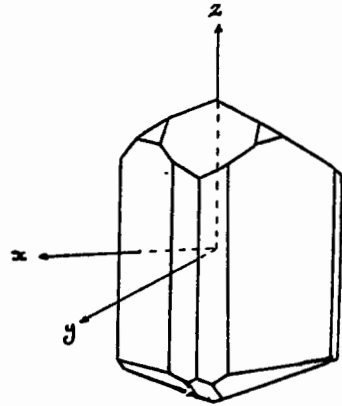


Fig. 5.

By the way, the relations between electric fields and strains are:

$$\left. \begin{aligned} e_{xx} &= -d_{22}E_y + d_{31}E_z, & e_{yz} &= d_{15}E_y, \\ e_{yy} &= d_{22}E_y + d_{31}E_z, & e_{zx} &= d_{15}E_z, \\ e_{zz} &= d_{33}E_z, & e_{xy} &= -2d_{22}E_x, \\ -d_{22} &= 0.69 \times 10^{-8} \text{ C.G.S. units}, & d_{31} &= 0.74 \times 10^{-8} \text{ C.G.S. units}, \\ d_{33} &= 5.78 \times 10^{-8} \text{ C.G.S. units}, & d_{15} &= 11.04 \times 10^{-8} \text{ C.G.S. units}. \end{aligned} \right\} (52)$$

(1) If a plate is cut by two planes perpendicular to the principal axis, the direction cosines of normal are

$$l = m = 0, \quad n = 1, \quad (53)$$

six quantities of (19) reduce to

$$\left. \begin{aligned} P^2 &= c_{44}, & PQ &= 0, \\ Q^2 &= c_{44}, & QR &= 0, \\ R^2 &= c_{33}, & PR &= 0. \end{aligned} \right\} (54)$$

Introducing these quantities into (9), we get

$$\left. \begin{aligned} c_1 = c_2 = c_{44} &= 66.71 \times 10^{10} \text{ dynes/cm}^2, \\ c_3 &= c_{33} = 160.69 \times 10^{10} \text{ dynes/cm}^2, \end{aligned} \right\} (55)$$

and from (7),

$$\left. \begin{aligned} \lambda_1 &= 1, & \mu_1 = \nu_1 &= 0, \\ \lambda_2 &= 1, & \nu_2 = \lambda_2 &= 0, \\ \lambda_3 &= 1, & \lambda_3 = \mu_3 &= 0. \end{aligned} \right\} (56)$$

The normal modes of vibrations are

$$\text{where } \left. \begin{aligned} \xi &= A \cos\left(q\pi \frac{z}{a}\right) \cdot e^{ipt}, \\ \xi_1 &= u, \\ \xi_2 &= v, \\ \xi_3 &= w, \end{aligned} \right\} \quad (57)$$

of which  $w$  can only be excited piezoelectrically by the electric field along the axis  $z$ , as is seen from (52). Accordingly the vibration is of pure longitudinal. The corresponding frequency of vibration is

$$\frac{p}{2\pi} = \frac{q}{2a} \sqrt{\frac{c_{33}}{\rho}} = \frac{q}{a} \times 0.360 \times 10^6 \text{ cycles/sec.} \quad (58)$$

(2) If a plate is cut by two planes perpendicular to the axis  $x$  all the theoretical results are quite similar to that of  $X$ -cut quartz plate. Thus

$$\left. \begin{aligned} c_1 &= c_{11} = 270.17 \times 10^{10} \text{ dynes/cm}^2, \\ c_2 &= 102.25 \times 10^{10} \text{ dynes/cm}^2, \\ c_3 &= 65.03 \times 10^{10} \text{ dynes/cm}^2, \end{aligned} \right\} \quad (59)$$

$$\left. \begin{aligned} \lambda_1 &= 1, & \mu_1 &= \nu_1 = 0, \\ \lambda_2 &= 0, & \mu_2 &= 0.977, & \nu_2 &= -0.213, \\ \lambda_3 &= 0, & \mu_3 &= -0.213, & \nu_3 &= -0.977. \end{aligned} \right\} \quad (60)$$

The normal modes of vibrations are

$$\text{where } \left. \begin{aligned} \xi &= A \cos\left(q\pi \frac{x}{a}\right) \cdot e^{ipt}, \\ \xi_1 &= u, \\ \xi_2 &= 0.977v - 0.213w, \\ \xi_3 &= -0.213v - 0.977w, \end{aligned} \right\} \quad (61)$$

while the electric field in the direction  $x$  cannot excite the displacement  $\xi_1$  and can hardly excite  $\xi_2$  as is seen from (52), and the period of  $\xi_2$  and  $\xi_3$  are :

$$\frac{p}{2\pi} = \frac{q}{2a} \sqrt{\frac{c_2}{\rho}} = \frac{q}{a} \times 0.287 \times 10^6 \text{ cycles/sec.}, \quad (62)$$

$$\frac{p}{2\pi} = \frac{q}{2a} \sqrt{\frac{c_3}{\rho}} = \frac{q}{a} \times 0.229 \times 10^6 \text{ cycles/sec.} \quad (63)$$

### 5. Conclusions.

1. The thickness vibration of a thin piezoelectric crystal plate is due to the standing wave produced by the interference of plane waves incident to and reflected from the major plane boundary surfaces of the plate.

2. There are three normal modes in the thickness vibration, and corresponding frequencies are given in the first approximation by a representative formula :

$$\frac{p}{2\pi} = \frac{q}{2a} \sqrt{\frac{c}{\rho}}, \quad (15)$$

where  $q$ —any integer,  
 $a$ —thickness of the plate,  
 $\rho$ —density of the medium,

and  $c$  is a root of the cubic equation

$$\begin{vmatrix} P^2 - c & PQ & PR \\ PQ & Q^2 - c & QR \\ PR & QR & R^2 - c \end{vmatrix} = 0, \quad (9)$$

where  $P = lc_1 + mc_8 + nc_5$ ,  
 $Q = lc_6 + mc_2 + nc_4$ ,  
 $R = lc_5 + mc_4 + nc_3$ ,

in which  $c_1, c_2, \dots$  have no quantitative meaning, but in  $P^2, PQ, \dots, c_1^2$  is to be replaced by  $c_{11}, c_{12}$  by  $c_{12}$  and so on,  $c_{11}, c_{12}, \dots$  being the adiabatic elastic constants.

Especially in Quartz and Tourmaline,

$$\left. \begin{aligned} P^2 &= l^2 c_{11} + \frac{1}{2} m^2 (c_{11} - c_{12}) + n^2 c_{44} + 2mnc_{14}, \\ Q^2 &= \frac{1}{2} l^2 (c_{11} - c_{12}) + m^2 c_{11} + n^2 c_{44} - 2mnc_{14}, \\ R^2 &= (l^2 + m^2) c_{44} + n^2 c_{33}, \\ PQ &= 2nlc_{14} + \frac{1}{2} lm(c_{11} + c_{12}), \\ QR &= (l^2 - m^2) c_{14} + mn(c_{13} + c_{44}), \\ PR &= 2lmc_{14} + nl(c_{13} + c_{44}). \end{aligned} \right\} \quad (19)$$

3. These three normal modes of vibration may or may not be sustained piezoelectrically according to the orientation of the major surfaces.

4. The fundamental frequency and mode of thickness vibration of quartz and tourmaline oscillating crystals are :

Cutting	Fundamental Frequency of Vibration	Mode of Vibration
Quartz {	X-cut $\frac{1}{2a} \sqrt{c_{11}/\rho}$	Longitudinal
	Y-cut $\frac{1}{2a} \sqrt{(c_{11}-c_{12})/2\rho}$	Shear
	R-cut $\frac{1}{2a} \sqrt{\{\frac{1}{2}(c_{11}-c_{12}) \sin^2\theta + c_{44} \cos^2\theta - c_{14} \sin 2\theta\}/\rho}$	Shear
	R'-cut $\frac{1}{2a} \sqrt{\{\frac{1}{2}(c_{11}-c_{12}) \sin^2\theta + c_{44} \cos^2\theta + c_{14} \sin 2\theta\}/\rho}$	Shear
Tourmaline Z-cut	$\frac{1}{2a} \sqrt{c_{33}/\rho}$	Longitudinal

where  $\theta = \arctan \frac{2.200}{\sqrt{3}}$  (37)

5. Electrically measured natural frequency of thickness vibration is always little higher than that calculated by the formula (15).

6. Adiabatic elastic constants of piezoelectric crystals are able to be determined in the first approximation by the measurement of frequencies of thickness vibrations of thin plates prepared with given crystals.

7. Many authors have employed the formula

$$\frac{p}{2\pi} = \frac{q}{2a} \sqrt{\frac{E}{\rho}} \quad (E: \text{Young's modulus})$$

for the frequency of longitudinal vibration of a thin plate, but it is obvious that this formula is only true provided the length  $a$  of the medium is sufficiently long compared with the dimensions of the cross section, and not valid for the thickness vibration as is also easily verified by numerical evaluation.

8. Harmonic vibration of quartz oscillating crystal plate can sometimes be maintained in the Pierce circuit.

9. It must be remembered that in our treatment the boundary surfaces are assumed to extend to infinity, so that the various complexities in frequency and mode of vibration observed by many authors do not enter into the discussion.