# Detection and Parameter Estimation in an Amplitude-Comparison Monopulse Radar

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*Abstract*—This paper considers the problem of detecting and estimating the unknown angular location of a radar target that is observed simultaneously by a number of antennas. The amplitude of the signal received by a particular antenna is assumed to depend on the angular location of the target, but the time of arrival of the signal is assumed to be the same at all of the antennas.

The generalized likelihood ratio test is used to derive the detection and parameter-estimation strategy for the radar receiver. Explicit expressions for the detector and the angle estimates are derived in a number of important special cases, and the performance of the detector is evaluated. Accuracy formulas for the angle estimates, valid for high signal-to-noise ratios, are derived for the general problem and used to compare the performance of an optimum four-beam monopulse system with a type of monopulse system commonly in use.

## Introduction

THIS PAPER is concerned with the problem of detection and parameter estimation in an amplitude-comparison monopulse radar. The term "amplitude comparison" refers to the fact that information about the angular location of a target in space is derived from a cluster of antenna beams whose gain patterns are skewed in angle so that the received signal appears in different antenna beams with amplitudes that depend on the target angle relative to the antenna complex. The antennas are so designed that the arrival times of the received waveform at the different antennas are nearly identical [1], [2].

The basic goal of this paper is to apply the generalized likelihood ratio test [3], [4] to derive a strategy for detecting targets whose amplitude, carrier phase, and angular location are unknown and to obtain estimates of these unknown parameters. The solution to this problem will be obtained and then extended to the case where additional target parameters such as range and velocity are unknown. The detection characteristic (probability of detection versus probability of false alarm) of the maximum-likelihood receiver will be derived in certain special cases, and expressions for the high signal-to-noise ratio covariance matrix of the angle estimates will be obtained.

There are actually several different versions of the problem described above, which depend on the physical situation at hand. First, the problem can be either one- or two-dimensional depending on whether the target has to be located in only one angular coordinate such as azimuth or elevation or whether it has to be located in both angular coordinates. In the first case a minimum of two beams is required, in the second case a minimum of three beams. Second, different mathematical problems arise depending on whether it is assumed that the target returns received on the various beams are coherent or incoherent with respect to one another. Solutions can be obtained for all of these versions of the problem, but the degree to which the solution can be made explicit and the degree to which the performance of the resulting receiver can be analyzed depend strongly on which version is under consideration. Fortunately, the analysis can be completed to a satisfactory degree in most cases of interest.

There does not seem to be a great deal of literature available on the problem considered in this paper. The most pertinent reference is a recent paper by McGinn [5], which discusses the estimation of a single target location angle and perhaps other unknown target parameters given two coherent receiver beams. The problems of detection, two-dimensional angle estimation, and incoherent receiver beams are not considered. Furthermore, the approach taken by the present paper does not require the beam-shape assumptions used by McGinn, and by explicitly treating carrier phase as an unknown parameter it obtains a number of new results.

Another pertinent reference is a paper by Urkowitz [6], which discusses the maximum angular estimation accuracy achievable by means of a physical antenna aperture. This is a more general problem than the one treated in the present paper in that Urkowitz considers antennas that are both phase and amplitude sensitive, but his accuracy formulas are not as explicit as those derived here for the special case of amplitude-sensitive antennas, and he does not attempt to derive the structure of the optimum receiver that achieves these accuracies. The problem of combined detection and angle estimation is not considered in his paper.

## Derivation of the Maximum-Likelihood Receiver

### *Incoherent Beams*

The case of incoherent beams will be considered first because the mathematics involved is simpler than in the coherent case. It is assumed that both angular coordinates of the target must be estimated and that $m$ beams, $m \geq 3$, are used. The complex envelopes $r_i(t)$ of the received waveforms are of the form

$$r_i(t) = Ae^{i\psi_i}G_i(\theta, \varphi)s(t) + n_i(t) \qquad i = 1, \cdots, m, \quad (1)$$

where $A \geq 0$ denotes the unknown received signal amplitude, $\psi_i$ the unknown carrier phase of the signal in the $i$th beam, and $s(t)$ the known complex envelope of the trans-

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mitted signal. For convenience this signal is normalized so that $\int |s(t)|^2\, dt = 1$. The $n_i$ denote the complex envelopes of zero-mean white Gaussian noises that are independent from channel to channel. In symbols,[1]

$$\begin{aligned} E[n_i(t)n_k^*(t')] &= 4N_0\, \delta_{ik}\, \delta(t - t') \\ E[n_i(t)n_k(t')] &= 0. \end{aligned} \tag{2}$$

The real function $G_i(\theta, \varphi)$ denotes the beam pattern of the $i$th beam measured with respect to the angular coordinates $\theta$ and $\varphi$. These coordinates may be thought of as azimuth and elevation angles, but any two other coordinates capable of defining a direction in space could be used. Thus, the term $AG_i(\theta, \varphi)$ denotes the amplitude of the signal received on the $i$th beam when the angular coordinates of the target are $\theta$ and $\varphi$. The assumption that the $G_i$ are real reflects the fact that the arrival times of the received waveform at the different antennas are assumed to be identical regardless of the angular location of the target. This is never exactly the case in a practical amplitude-comparison monopulse system, but careful antenna design can yield a system for which it is a reasonable assumption.

The beam-to-beam incoherence of the problem comes about through the assumption that the carrier phases $\psi_i$ are unknown and not necessarily the same in each channel. When the coherent case is treated, a single common unknown carrier phase will be postulated.

Stated in mathematical language, the problem at hand is to observe the time waveforms $r_i(t)$, $i = 1, \cdots, m$ in the observation interval[2] $0 \leq t \leq T$ and then to decide whether a target is present or not ($A \geq 0$ or $A = 0$). If the decision is "target present," estimates of the unknown parameters, $A$, $\psi_i$, $\theta$, and $\varphi$ are to be obtained. The generalized likelihood ratio test will be used to perform this task. This test requires the receiver to calculate the quantity[3]

$$L = \frac{\max\limits_{A, \boldsymbol{\psi}, \theta, \varphi} p[\mathbf{r}(t), 0 \leq t \leq T \mid A, \theta, \varphi, \boldsymbol{\psi}]}{p[\mathbf{r}(t), 0 \leq t \leq T \mid A = 0]}, \tag{3}$$

where $\mathbf{r}(t)$ denotes the set of values $r_1(t), \cdots, r_m(t)$ and $\boldsymbol{\psi}$ denotes the set of values $\psi_1, \cdots, \psi_m$, and to compare it with a present threshold $\lambda$. If $L < \lambda$, the decision "target absent" is made. If $L > \lambda$, "target present" is announced and the values of $A$, $\psi_i$, $\theta$, and $\varphi$ that achieved the maximum in (3) are taken as estimates of the true values of these parameters. Making use of (1) and (2), (3) can be written

$$L = \frac{\max\limits_{A, \boldsymbol{\psi}, \theta, \varphi} \exp\left\{-\frac{1}{4N_0} \sum_{i=1}^{m} \int_0^T |r_i(t) - Ae^{i\psi_i}G_i(\theta, \varphi)s(t)|^2\, dt\right\}}{\exp\left\{-\frac{1}{4N_0} \sum_{i=1}^{m} \int_0^T |r_i(t)|^2\, dt\right\}}, \tag{4}$$

which can be simplified with the result,

$$\begin{aligned} l &\equiv 4N_0 \ln L \\ &= \max_{A, \boldsymbol{\psi}, \theta, \varphi} \sum_{i=1}^{m} \{2AG_i(\theta, \varphi) \operatorname{Re}[y_i e^{-i\psi_i}] - A^2G_i^2(\theta, \varphi)\}, \end{aligned} \tag{5}$$

where

$$y_i = \int_0^T r_i(t)s^*(t)\, dt. \tag{6}$$

The complex voltages $y_i$ can be formed by passing $r_i(t)$ through a filter matched to the known waveform $s(t)$.

The maximization on $A$ is accomplished by differentiating the summation appearing in (5) with respect to $A$ and setting this derivative equal to zero. The resulting equation is

$$\sum_{i=1}^{m} \{2G_i(\theta, \varphi) \operatorname{Re}[y_i e^{-i\psi_i}] - 2AG_i^2(\theta, \varphi)\} = 0$$

or

$$A = \frac{\sum_{i=1}^{m} G_i(\theta, \varphi) \operatorname{Re}[y_i e^{-i\psi_i}]}{\sum_{i=1}^{m} G_i^2(\theta, \varphi)}. \tag{7}$$

Substitution of (7) in (5) yields the result

$$l = \max_{\boldsymbol{\psi}, \theta, \varphi} \frac{\left[\sum_{i=1}^{m} G_i(\theta, \varphi) \operatorname{Re}(y_i e^{-i\psi_i})\right]^2}{\sum_{i=1}^{m} G_i^2(\theta, \varphi)}. \tag{8}$$

The next step is to perform the maximization with respect to the parameters $\boldsymbol{\psi}$. Inspection of (8) shows that it is maximized when

$$\psi_i = \arg[y_i G_i(\theta, \varphi)]$$

and this maximum is given by

$$l = \max_{\theta, \varphi} \frac{\left[\sum_{i=1}^{m} |y_i|\, |G_i(\theta, \varphi)|\right]^2}{\sum_{i=1}^{m} G_i^2(\theta, \varphi)}. \tag{9}$$

The next step would be to maximize with respect to $\theta$ and $\varphi$; however, no way of explicitly performing this operation has been found when $m$ is greater than 3. To see the reason for this, note that the Schwarz inequality applied to (9) yields

$$l = \max_{\theta, \varphi} \frac{\left[\sum_{i=1}^{m} |y_i|\, |G_i(\theta, \varphi)|\right]^2}{\sum_{i=1}^{m} G_i^2(\theta, \varphi)} \leq \sum_{i=1}^{m} |y_i|^2 \tag{10}$$

and furthermore, equality is achieved in this bound, $l = \sum_{i=1}^{m} |y_i|^2$ if and only if

$$|G_i(\theta, \varphi)| = \alpha\, |y_i| \qquad i = 1, \cdots, m, \tag{11}$$

[1] The asterisk denotes complex conjugation. This complex noise corresponds to a real white noise with a two-sided power density of $N_0$ watts per cycle.

[2] $T$ is assumed to be much larger than the duration of $s(t)$.

[3] Strictly speaking, neither numerator nor denominator of this expression exists. But, Grenander has shown that it is possible to define a suitable likelihood ratio which approaches the quantity given by (3) in the limit as the noises involved become white.

where $\alpha$ denotes an arbitrary proportionality constant. If $m = 3$, (11) can be satisfied by choosing $\hat{\theta}$ and $\hat{\varphi}$ such that

$$\left|\frac{G_1(\hat{\theta}, \hat{\varphi})}{G_3(\hat{\theta}, \hat{\varphi})}\right| = \frac{|y_1|}{|y_3|}, \qquad \left|\frac{G_2(\hat{\theta}, \hat{\varphi})}{G_3(\hat{\theta}, \hat{\varphi})}\right| = \frac{|y_2|}{|y_3|} \tag{12}$$

and then choosing $\alpha = |G_3(\hat{\theta}, \hat{\varphi})|/|y_3|$. Examples of gain functions $G_i$ can be constructed for which (12) cannot be solved for $\hat{\theta}$, $\hat{\varphi}$ for all possible $|y_i|$. Therefore, it will be necessary to assume that the functions $G_i$ are of such a form that the equations above can be solved for $\hat{\theta}$, $\hat{\varphi}$ for all non-negative values on their right-hand sides. This assumption will be in force, wherever it is applicable, throughout the remainder of the paper.

When $m > 3$, there are more equations than unknowns and no solution to (11) will exist in general. In this case,

$$l \neq \sum_{i=1}^{m} |y_i|^2$$

and no explicit way of performing the maximization required by (9) has been found.

The results for the case $m = 3$ can be summarized by saying that the maximum-likelihood strategy leads to a receiver that computes the quantity

$$l = |y_1|^2 + |y_2|^2 + |y_3|^2 \tag{13}$$

and compares it with a threshold. If $l < \lambda$, "target absent" is announced. If $l > \lambda$, "target present" is announced, and the values of $\hat{\theta}$ and $\hat{\varphi}$ that solve (12) are taken as estimates of the true values of these parameters.

When only one target angle is unknown or when fan beams are used so that the antenna gains are only a function of one target angle, then a minimum of two beams must be used to estimate the unknown angle. By an analysis that is identical, almost word for word, to the one just given, it can be shown that, for $m = 2$, the receiver compares the quantity

$$l = |y_1|^2 + |y_2|^2 \tag{14}$$

with a threshold and uses the value of $\theta$ satisfying

$$\left|\frac{G_1(\hat{\theta})}{G_2(\hat{\theta})}\right| = \frac{|y_1|}{|y_2|} \tag{15}$$

as the estimate of $\theta$. Once again, it will be assumed that the $G_i$ are of such a form that (15) has a solution for all non-negative values of its right-hand side.

Even in those cases where it is not possible to give an explicit expression for $l$, it is still possible to give an interesting and useful physical interpretation of this quantity. The crux of this interpretation lies in the fact that a linear combination of the beam voltages $|y_i|$ results in a synthetic beam pointing in some direction in between the directions of the original beams. More precisely, if there is no system noise,

$$\sum_{i=1}^{m} |y_i|\,|G_i(\theta, \varphi)| = A \sum_{i=1}^{m} |G_i(\theta_0, \varphi_0)G_i(\theta, \varphi)|, \tag{16}$$

where $\theta_0$, $\varphi_0$ denote the true angular coordinates of the target. The right-hand side of (16), considered as a function of $\theta_0$, $\varphi_0$ for fixed $\theta$, $\varphi$ has its maximum at $\theta_0 = \theta$, $\varphi_0 = \varphi$. Thus, it represents a beam pointing in the direction defined by the angular coordinates $\theta$, $\varphi$. The shape of this beam depends, of course, on the detailed shapes of the original beam patterns $G_i$. It is now obvious that the quantity $l$ is the maximum response that results when the synthetic beam is steered to all possible directions in search of the target. (The denominator of (9) is simply a normalization factor that forces the maximum gains of all the synthetic beams to be equal.)

*Coherent Beams*

In this case, the complex envelopes of the signals received on the $m$ beams are given by the expressions

$$r_i(t) = Ae^{i\psi}G_i(\theta, \varphi)s(t) + n_i(t) \qquad i = 1, \cdots, m,$$

where the meaning of all symbols is the same as in the incoherent case. The only difference between the coherent and incoherent cases is that the carrier phase angle $\psi$, although unknown, is assumed to be the same in all receiver beams.

The likelihood ratio $L$ can be derived using the same techniques as in the incoherent case, with the result

$$l \equiv 4N_0 \ln L$$

$$= \max_{A,\psi,\theta,\varphi} \sum_{i=1}^{m} \{2AG_i(\theta, \varphi) \operatorname{Re}[y_i e^{-i\psi}] - A^2G_i^2(\theta, \varphi)\}. \tag{17}$$

The maximization on $A$ proceeds exactly as before and leads to

$$l = \max_{\psi,\theta,\varphi} \frac{\left\{\sum_{i=1}^{m} G_i(\theta, \varphi) \operatorname{Re}[y_i e^{-i\psi}]\right\}^2}{\sum_{i=1}^{m} G_i^2(\theta, \varphi)} \tag{18}$$

Upon rewriting (18) as follows,

$$l = \max_{\psi,\theta,\varphi} \frac{\left\{\operatorname{Re}\left[\sum_{i=1}^{m} y_i G_i(\theta, \varphi)e^{-i\psi}\right]\right\}^2}{\sum_{i=1}^{m} G_i^2(\theta, \varphi)}, \tag{19}$$

it is seen readily that its maximum with respect to $\psi$ occurs for

$$\psi = \arg\left[\sum_{i=1}^{m} y_i G_i(\theta, \varphi)\right] \tag{20}$$

and is given by

$$l = \max_{\theta,\varphi} \frac{\left|\sum_{i=1}^{m} y_i G_i(\theta, \varphi)\right|^2}{\sum_{i=1}^{m} G_i^2(\theta, \varphi)}. \tag{21}$$

The expression for $l$ given by (21) can be given a synthetic beam interpretation similar to the one given for the incoherent problem. In fact, the only difference between the two interpretations is that in the coherent case the synthetic beams are formed at RF; in the incoherent case they were formed at video.

The analysis so far has differed very little from that performed for the incoherent case. The next step is to perform the maximization on $\theta$ and $\varphi$ indicated in (21), and it is at this step that a real difference between the two cases arises. In the incoherent case, the Schwarz inequality was used for this purpose, but in the coherent case the upper bound so derived cannot be achieved because the $G_i$ are real whereas the $y_i$ are complex.

In one special case (two beams measuring one angle) it is possible to perform the required maximization directly, but this process is quite tedious and it seems more fruitful to attack the problem from another viewpoint.

The starting point for this approach is the expression for $l$ given by (18), from which the carrier phase angle $\psi$ has not yet been eliminated. Application of the Schwarz inequality to this expression yields the bound

$$l \leq \max_{\psi} \sum_{i=1}^{m} \{\mathrm{Re}\,[y_i e^{-j\psi}]\}^2, \tag{22}$$

which can be achieved if and only if

$$G_i(\theta, \varphi) = \alpha \,\mathrm{Re}\,[y_i e^{-j\psi}] \qquad i = 1, \cdots, m, \tag{23}$$

where $\alpha$ denotes an arbitrary proportionality constant. This bound can be rewritten in the form

$$l \leq \tfrac{1}{2} \sum_{i=1}^{m} |y_i|^2 + \max_{\psi} \tfrac{1}{2}\,\mathrm{Re}\left[e^{-2j\psi} \sum_{i=1}^{m} y_i^2\right] \tag{24}$$

by using the identity $\mathrm{Re}\, x\, \mathrm{Re}\, y = \frac{1}{2}\mathrm{Re}\, xy^* + \frac{1}{2}\mathrm{Re}\, xy$.

The maximum required by (24) is obviously achieved when $\psi = \hat{\psi}$ where

$$\hat{\psi} = \tfrac{1}{2} \arg\left[\sum_{i=1}^{m} y_i^2\right], \tag{25}$$

which results in the following expression for the upper bound on $l$,

$$l \leq \tfrac{1}{2} \sum_{i=1}^{m} |y_i|^2 + \tfrac{1}{2}\left|\sum_{i=1}^{m} y_i^2\right|. \tag{26}$$

To recapitulate, $l$ is bounded above as indicated in (26), and this upper bound is achieved if and only if (23) is satisfied with $\psi = \hat{\psi}$ as given by (25). These conditions cannot be met in general if $m \geq 3$ because there are then more equations than unknowns. When $m = 3$, the equations can be solved by choosing $\hat{\theta}$, $\hat{\varphi}$ so that

$$\frac{G_1(\hat{\theta}, \hat{\varphi})}{G_3(\hat{\theta}, \hat{\varphi})} = \frac{\mathrm{Re}\,[y_1 e^{-j\hat{\psi}}]}{\mathrm{Re}\,[y_3 e^{-j\hat{\psi}}]}; \quad \frac{G_2(\hat{\theta}, \hat{\varphi})}{G_3(\hat{\theta}, \hat{\varphi})} = \frac{\mathrm{Re}\,[y_2 e^{-j\hat{\psi}}]}{\mathrm{Re}\,[y_3 e^{-j\hat{\psi}}]}, \tag{27}$$

where $\hat{\psi}$ is given by (25), and then setting $\alpha = G_3(\hat{\theta}, \hat{\varphi})/\mathrm{Re}\,[y_3 e^{-j\hat{\psi}}]$. When (27) can be satisfied, $l$ is given by the expression

$$l = \tfrac{1}{2}[|y_1|^2 + |y_2|^2 + |y_3|^2] + \tfrac{1}{2}\,|y_1^2 + y_2^2 + y_3^2|. \tag{28}$$

It will be necessary to assume that the gains $G_i$ are of such a form that (27) can be solved for $\hat{\theta}$, $\hat{\varphi}$ for all real values of their right-hand sides.

The expression for $l$ given by (28) has an interesting interpretation. It is a mixture of an incoherent combination of the beam voltages as given by the first term of the expression and a coherent combination given by the second term. The coherent combination is not the usual one ($|y_1 + y_2 + y_3|^2$), but rather a coherent combination of squares of the beam voltages. This has the effect of weighting the larger beam voltages more heavily than the smaller ones. It may seem surprising at first that a straightforward coherent combination of the beam voltages is not the right thing to do. The reason for this is that the amplitude of the target return is in general not the same in each of the beams. When this is the case, coherent combination can yield a significantly lower output signal-to-noise ratio than incoherent combination.[4]

The detection function given by (28) and the angle estimates given by (27) are both considerably more difficult to realize than their counterparts in the incoherent case. The source of this difficulty is the term $|y_1^2 + y_2^2 + y_3^2|$. Several attempts have been made to rewrite the pertinent expressions in a form not containing such a term, but so far no success has been achieved except in an important special case. When attention is restricted to the case of two beams measuring a single target angle, it is possible to write the solution in a form that is both elegant and relatively easy to realize.

The same kind of analysis given above leads to the solution of the two-beam problem as expressed by

$$l = \tfrac{1}{2}[|y_1|^2 + |y_2|^2] + \tfrac{1}{2}\,|y_1^2 + y_2^2| \tag{29}$$

$$\frac{G_1(\hat{\theta})}{G_2(\hat{\theta})} = \frac{\mathrm{Re}\,[y_1 e^{-j\hat{\psi}}]}{\mathrm{Re}\,[y_2 e^{-j\hat{\psi}}]} \tag{30}$$

$$\hat{\psi} = \tfrac{1}{2} \arg\,(y_1^2 + y_2^2). \tag{31}$$

This solution is based on the assumption that

$$G_1(\theta)/G_2(\theta) = x$$

can be solved for $\theta$, given any real value of $x$. These equations can be simplified by introducing the following change of variables[5]

$$\begin{aligned} \eta_1 &= y_1 + jy_2 \\ \eta_2 &= y_1 - jy_2. \end{aligned} \tag{32}$$

Substitution of (32) into (29) yields

$$\begin{aligned} l &= \tfrac{1}{2}\left\{|\tfrac{1}{2}(\eta_1 + \eta_2)|^2 + \left|\frac{1}{2j}(\eta_1 - \eta_2)\right|^2 + |\eta_1\eta_2|\right\} \\ &= \tfrac{1}{2}\{\tfrac{1}{2}\,|\eta_1|^2 + \tfrac{1}{2}\,|\eta_2|^2 + |\eta_1\eta_2|\} \\ &= \tfrac{1}{4}[|\eta_1| + |\eta_2|]^2. \end{aligned} \tag{33}$$

[4] The authors are grateful to J. Sawyers of the Hughes Aircraft Company, Fullerton, Calif., for pointing this fact out to them.

[5] The authors are indebted to J. Margolin of Lincoln Laboratory for suggesting this change of variables.

Similarly, substitution of (32) into (30) and (31) yields[6]

$$\exp(j\hat{\psi}) = \left(\frac{\eta_1\eta_2}{|\eta_1\eta_2|}\right)^{1/2} \tag{34}$$

and

$$\begin{aligned}\frac{G_1(\hat{\theta})}{G_2(\hat{\theta})} &= \frac{\mathrm{Re}\,[(\eta_1+\eta_2)(\eta_1^*\eta_2^*)^{1/2}]}{\mathrm{Re}\left(\frac{1}{j}(\eta_1-\eta_2)(\eta_1^*\eta_2^*)^{1/2}\right)}\\ &= \frac{\mathrm{Re}\,[|\eta_1|\,(\eta_1\eta_2^*)^{1/2}+|\eta_2|\,(\eta_1^*\eta_2)^{1/2}]}{\mathrm{Im}\,[|\eta_1|\,(\eta_1\eta_2^*)^{1/2}-|\eta_2|\,(\eta_1^*\eta_2)^{1/2}]}\\ &= \frac{(|\eta_1|+|\eta_2|)\,\mathrm{Re}\,(\eta_1\eta_2^*)^{1/2}}{(|\eta_1|+|\eta_2|)\,\mathrm{Im}\,(\eta_1\eta_2^*)^{1/2}}\\ &= \cot\tfrac{1}{2}(\arg\eta_1-\arg\eta_2). \end{aligned}\tag{35}$$

The realization of the two-beam receiver is now very simple. The voltages $\eta_1$ and $\eta_2$ can be obtained from the matched filter outputs $y_1$ and $y_2$ by means of a hybrid junction. The detection function $l$ is then formed with linear envelope detectors and an adder, and a voltage proportional to $2\cot^{-1} G_1(\hat{\theta})/G_2(\hat{\theta})$ is formed by combining $\eta_1$ and $\eta_2$ in a phase detector. This angle estimation scheme is very similar to some of those described in [1] and [2].

One further point is worthy of mention in connection with the problem just considered. Many practical two-beam monopulse systems do not work directly with the signals $y_1$ and $y_2$, but rather with the sum and difference signals $y_\Sigma = y_1 + y_2$ and $y_\Delta = y_1 - y_2$. The antenna gain functions associated with these signals are $G_\Sigma(\theta) = G_1(\theta) + G_2(\theta)$ and $G_\Delta(\theta) = G_1(\theta) - G_2(\theta)$, respectively. Since it is easily verified that

$$\frac{|y_1G_1(\theta)+y_2G_2(\theta)|^2}{G_1^2(\theta)+G_2^2(\theta)} = \frac{|y_\Delta G_\Delta(\theta)+y_\Sigma G_\Sigma(\theta)|^2}{G_\Delta^2(\theta)+G_\Sigma^2(\theta)},$$

it follows that the maximum-likelihood receiver based on $y_\Sigma$ and $y_\Delta$ performs exactly as well as the receiver based on $y_1$ and $y_2$ and, in addition, has exactly the same structure. The advantage of this sum-and-difference beam approach is that for many monopulse designs the function $G_\Delta(\theta)/G_\Sigma(\theta)$ is approximately linear in $\theta$ for $\theta$ in the range of interest. This fact greatly simplifies the circuitry needed to implement the estimation procedure.

### *Additional Unknown Parameters*

In this section it will be assumed that the signal waveform depends on several other parameters, denoted by the vector $\alpha$, in addition to those already considered. Thus, the received signals are of the form

$$r_i(t) = Ae^{i\psi_i}G_i(\theta,\varphi)s(t,\alpha) + n_i(t) \qquad i = 1,\cdots,m \tag{36}$$

in the incoherent case and of the form

$$r_i(t) = Ae^{i\psi}G_i(\theta,\varphi)s(t,\alpha) + n_i(t) \qquad i = 1,\cdots,m \tag{37}$$

in the coherent case. It is still assumed that the signal is normalized so that $\int |s(t,\alpha)|^2\,dt = 1$.

A few moments' thought will reveal that the only effect the additional parameters will have on the solutions already obtained is to replace the old $y_i$ by

$$y_i(\alpha) = \int_0^T r_i(t)s^*(t,\alpha)\,dt \tag{38}$$

and to require that the new $l$ be obtained from the old $l = l(\alpha)$ by maximizing the latter on $\alpha$. For example, in the incoherent, two-beam, one-angle problem,

$$l = \max_\alpha\,[|y_1(\alpha)|^2 + |y_2(\alpha)|^2] \tag{39}$$

$$\frac{G_1(\hat{\theta})}{G_2(\hat{\theta})} = \frac{|y_1(\hat{\alpha})|}{|y_2(\hat{\alpha})|}, \tag{40}$$

where $\hat{\alpha}$ denotes the value of $\alpha$ that achieves the maximum in (39). The reader should experience no difficulty in extending all the other results of the previous section to the case where additional signal parameters are unknown.

The additional maximization on $\alpha$ that is required when calculating $l$ often cannot be performed explicitly. When this is the case, the usual solution is to use a bank of $N$-filters matched to $s(t,\alpha_1),\cdots,s(t,\alpha_N)$ to simultaneously calculate the quantities $l(\alpha_1),\cdots,l(\alpha_N)$. The largest one of these values is taken as the desired value of $l$, and the $\alpha_i$ for which it occurs is used as the estimate of $\alpha$. This technique will give satisfactory results as long as the values of the $\alpha_i$ are sufficiently close together to give a reasonably accurate picture of the function $l(\alpha)$. Naturally, if the number of unknown parameters is large, this method may become too unwieldy to realize in practice.

A concrete example of the procedure just described arises when the additional unknown parameters are range delay $\tau$ and Doppler frequency $f$, i.e.,

$$s(t,\alpha) = s(t-\tau)e^{i2\pi ft}. \tag{41}$$

The quantity $y_i(\tau, f)$ can be formed by passing $r_i(t)$ through a filter whose impulse response is $s^*(-t)e^{i2\pi tf}$ and then sampling the output at time $\tau$. A finite bank of such filters spaced by the Doppler resolution of $s(t)$ and spanning the range of expected Doppler frequencies often will yield a close enough approximation to $y_i(\tau, f)$. The operation of the receiver for, say, the incoherent two-beam, one-angle problem now can be described. The outputs from the two beams are processed in two identical filter banks of the type just considered. The outputs from the corresponding Doppler channels are detected in square-law detectors, added, and then compared with a threshold. Whenever this threshold is exceeded, "target present" is announced. The time of the peak response following the threshold crossing is the estimated range delay, and the frequency of the Doppler channel having this peak response is the estimated Doppler frequency. The ratio of the outputs of the square-law detectors is formed and used to obtain the estimate of target angle according to (40).

[6] Either square root may be used.

## Receiver Performance

### *Receiver-Detection Characteristic*

The receiver-detection characteristic (detection probability $P_D$ versus false-alarm probability $P_F$ or information equivalent to this) has been calculated for the two-beam, one-angle and three-beam, two-angle incoherent receivers and for the two-beam, one-angle coherent receiver. No analytical way of performing this calculation for any of the remaining cases has been found.

The detection functions for the incoherent two-beam and three-beam receivers are given by (14) and (13), respectively. These equations represent the result of summing the quadratically detected envelopes of the pertinent matched filter outputs. The probability distributions of such functions are quite easy to calculate. As the details of this calculation are readily available in the literature [7], [8], only the results will be given here. The false-alarm probability $P_F$ is given by

$$P_F = \exp\left(-\frac{\lambda}{4N_0}\right)\sum_{k=0}^{m-1}\frac{1}{k!}\left(\frac{\lambda}{4N_0}\right)^k, \tag{42}$$

where $\lambda$ is the threshold setting and $m$ denotes the number of beams in use. The detection probability $P_D$ is given by

$$P_D = Q_m(\sqrt{E/N_0}, \sqrt{\lambda/2N_0}), \tag{43}$$

where

$$Q_m(\alpha, \beta) = \int_\beta^\infty x\left(\frac{x}{\alpha}\right)^{m-1}\exp\left(-\tfrac{1}{2}(x^2+\alpha^2)\right)I_{m-1}(\alpha x)\,dx \tag{44}$$

denotes the generalized $Q$ function and $E$ is the received signal energy,

$$E = \frac{A^2}{2}\sum_{k=1}^{2}G_k^2(\theta)$$

or

$$E = \frac{A^2}{2}\sum_{k=1}^{3}G_k^2(\theta, \varphi) \tag{45}$$

in the two- and three-beam cases, respectively.

The generalized $Q$ function defined by (44) is closely related to the incomplete Toronto function which has been plotted extensively by Marcum [7]. The use of these curves in conjunction with (42) through (45) makes it possible to construct curves of $P_D$ versus $P_F$ for any desired signal-to-noise ratio $E/N_0$.

The detection function for the two-beam coherent receiver is given by (29) or, equivalently, (33). Equation (33) represents the summation of two linearly detected RF voltages $\eta_1$ and $\eta_2$ of the form

$$\begin{aligned}\eta_1 &= A[G_1(\theta) + jG_2(\theta)]e^{j\psi} + \xi_1\\ \eta_2 &= A[G_1(\theta) - jG_2(\theta)]e^{j\psi} + \xi_2,\end{aligned} \tag{46}$$

where

$$\begin{aligned}\xi_1 &= \int [n_1(t) + jn_2(t)]s^*(t)\,dt\\ \xi_2 &= \int [n_1(t) - jn_2(t)]s^*(t)\,dt.\end{aligned} \tag{47}$$

Note that $\xi_1$ and $\xi_2$ are statistically independent, identically distributed, complex Gaussian voltages and note also that the means of $\eta_1$ and $\eta_2$ have the same magnitude,

$$|E(\eta_1)| = |E(\eta_2)| = A[G_1^2(\theta) + G_2^2(\theta)]^{1/2}. \tag{48}$$

Thus, the quantity $|\eta_1| + |\eta_2|$ is the sum of two linearly detected RF voltages consisting of equal-amplitude RF signals to which statistically independent, identically distributed Gaussian noises have been added. The problem of calculating the probability distribution of such a sum was attacked by Marcum [7] who, although unable to obtain a closed-form solution, showed that the difference in performance obtained by adding linearly detected voltages as compared to quadratically detected voltages is negligible for practical purposes. The conclusion to be drawn from this result is that the detection function $|\eta_1|^2 + |\eta_2|^2$ will perform almost as well as the optimum detection function $|\eta_1| + |\eta_2|$. But

$$|\eta_1|^2 + |\eta_2|^2 = 2[|y_1|^2 + |y_2|^2],$$

and it follows that the two-beam coherent detector performs about as well as the two-beam incoherent detector that has already been analyzed.

### *Angular Accuracy*

The problem of calculating exact expressions for the means and variances of the angle estimates obtained above is a formidable one. On the other hand, it is relatively easy to obtain approximate expressions for these quantities valid for high signal-to-noise ratios. This has been done by Swerling [9] for the case of a real scalar signal masked by additive Gaussian noise. It is a straightforward task to extend his results to the case where the complex envelopes of several waveforms are observed simultaneously. With this generalization, Swerling's results read as follows.

Given the complex envelopes $r_i(t)$, $i = 1, \cdots, m$ of the form

$$r_i(t) = s_i(t;\alpha) + n_i(t) \qquad i = 1, \cdots, m, \tag{49}$$

where $s_i(t;\alpha)$ is the complex envelope of a waveform that is known except for a finite number of unknown parameters described by the vector $\alpha$ and where the noises $n_i(t)$ have the properties described by (2) then, for large signal-to-noise ratios, the maximum-likelihood estimate $\hat{\alpha}$ of $\alpha$ is unbiased,

$$E(\hat{\alpha}) = \alpha \tag{50}$$

and has the covariance matrix

$$\begin{aligned}\Lambda &\equiv E[(\hat{\alpha} - \alpha)(\hat{\alpha} - \alpha)']\\ &= 2N_0\left[\sum_{i=1}^{m}\mathrm{Re}\int\frac{\partial s_i(t;\alpha)}{\partial\alpha}\left(\frac{\partial s_i(t;\alpha)}{\partial\alpha}\right)^* dt\right]^{-1}\end{aligned} \tag{51}$$

In (51), the prime denotes the matrix transpose, the asterisk denotes conjugate-matrix transpose, and $\partial f(x)/\partial x$ denotes a column vector whose $k$th component is $\partial f(x)/\partial x_k$. The covariance matrix given by the right-hand side of (51) is the same as the Cramér–Rao [9], [10] lower bound for the covariance matrix of any unbiased estimator of $\alpha$; therefore, it follows that for large signal-to-noise ratios the maximum-likelihood estimate of $\alpha$ is optimum in the sense that no other estimate can have a smaller covariance matrix.[7]

This general result now will be specialized to the monopulse problems considered earlier. For the $m$-beam coherent system, the parameter vector is given by $\alpha = [A, \psi, \theta, \varphi]$ and

$$s_i(t; \alpha) = Ae^{i\psi}G_i(\theta, \varphi)s(t). \tag{52}$$

Making these substitutions in (51) leads to the following asymptotic covariance matrix,

$$\mathbf{\Lambda} = 2N_0 \cdot \left[\begin{array}{cc|cc} ||\mathrm{G}||^2 & 0 & A(\mathrm{G}, \mathrm{G}_\theta) & A(\mathrm{G}, \mathrm{G}_\varphi) \\ 0 & A^2\,||\mathrm{G}||^2 & 0 & 0 \\ \hline A(\mathrm{G}, \mathrm{G}_\theta) & 0 & A^2\,||\mathrm{G}_\theta||^2 & A^2(\mathrm{G}_\theta, \mathrm{G}_\varphi) \\ A(\mathrm{G}, \mathrm{G}_\varphi) & 0 & A^2(\mathrm{G}_\theta, \mathrm{G}_\varphi) & A^2\,||\mathrm{G}_\varphi||^2 \end{array}\right], \tag{53}$$

where $\mathrm{G}' = [G_1(\theta, \varphi), \cdots, G_m(\theta, \varphi)]$, $(\mathrm{x}, \mathrm{y})$ denotes the dot product of the vectors x and y and $||\mathrm{x}||^2 = (\mathrm{x}, \mathrm{x})$.

This matrix can be inverted by making use of Frobenius' relation for the inverse of a partitioned matrix [11]. This relation states that

$$\left[\begin{array}{c|c} \mathrm{A}_{11} & \mathrm{A}_{12} \\ \hline \mathrm{A}_{21} & \mathrm{A}_{22} \end{array}\right]^{-1} = \left[\begin{array}{c|c} \mathrm{A}_{11}^{-1} + \mathrm{A}_{11}^{-1}\mathrm{A}_{12}\,\mathbf{\Delta}^{-1}\mathrm{A}_{21}\mathrm{A}_{11}^{-1} & -\mathrm{A}_{11}^{-1}\mathrm{A}_{12}\,\mathbf{\Delta}^{-1} \\ \hline -\mathbf{\Delta}^{-1}\mathrm{A}_{21}\mathrm{A}_{11}^{-1} & \mathbf{\Delta}^{-1} \end{array}\right], \tag{54}$$

where

$$\mathbf{\Delta} = \mathrm{A}_{22} - \mathrm{A}_{21}\mathrm{A}_{11}^{-1}\mathrm{A}_{12}. \tag{55}$$

If $\mathbf{\Lambda}$ is partitioned as shown in (53), it then follows that the covariance matrix for the angle estimates alone is given by $\Delta^{-1}$. Making the necessary substitutions now leads to the result,

$$\begin{aligned} \mathbf{\Lambda}_{\theta\varphi} &\equiv E[(\hat{\theta} - \theta, \hat{\varphi} - \varphi)'(\hat{\theta} - \theta, \hat{\varphi} - \varphi)] \\ &= \frac{1}{\rho}\begin{bmatrix} ||\mathrm{g}_\theta||^2 & (\mathrm{g}_\theta, \mathrm{g}_\varphi) \\ (\mathrm{g}_\theta, \mathrm{g}_\varphi) & ||\mathrm{g}_\varphi||^2 \end{bmatrix}^{-1}, \end{aligned} \tag{56}$$

where the normalized gain vector $\mathrm{g} = \mathrm{g}(\theta, \varphi)$ is given by

$$\mathrm{g} = ||\mathrm{G}||^{-1}\,\mathrm{G}, \tag{57}$$

and the signal-to-noise ratio $\rho$ is given by

$$\rho = (A^2/2N_0)\,||\mathrm{G}||^2. \tag{58}$$

Some insight into the reason why the particular normalization defined by (57) arises in this problem can be obtained by noting that the received waveforms can be written in the form

$$r_i(t) = A\,e^{i\psi}\,||\mathrm{G}||\,g_i(\theta, \varphi)s(t) + n_i(t). \tag{59}$$

Thus, this normalization divides the effect of the target location into two parts. The first is the term $||\mathrm{G}||$, which is the same in all the beams and represents a modification of the signal amplitude $A$ that is brought about by the target location. The second is the term $g_i(\theta, \varphi)$, which varies from beam to beam and gives a measure of the extent to which target location produces differential changes in the signal amplitude received by the various beams. Viewed in this light, it seems natural that the first term should affect the angular accuracy only through the signal-to-noise ratio and that all the other factors influencing angular accuracy should be expressible in terms of the normalized gain functions $g_i$. This is exactly what is expressed by (56).

The matrix appearing in (56) can be written in the form

$$\Lambda_{\theta,\varphi} = \frac{1}{\rho D}\begin{bmatrix} ||\mathrm{g}_\varphi||^2 & -(\mathrm{g}_\theta, \mathrm{g}_\varphi) \\ -(\mathrm{g}_\theta, \mathrm{g}_\varphi) & ||\mathrm{g}_\theta||^2 \end{bmatrix}, \tag{60}$$

where

$$D = ||\mathrm{g}_\theta||^2\,||\mathrm{g}_\varphi||^2 - (\mathrm{g}_\theta, \mathrm{g}_\varphi)^2. \tag{61}$$

The variances of $\hat{\theta}$ and $\hat{\varphi}$, as given by (60), have an interesting geometrical interpretation. Noting the simple identity,

$$\frac{D}{||\mathrm{g}_\varphi||^2} = \left|\left|\mathrm{g}_\theta - \left(\mathrm{g}_\theta, \frac{\mathrm{g}_\varphi}{||\mathrm{g}_\varphi||}\right)\frac{\mathrm{g}_\varphi}{||\mathrm{g}_\varphi||}\right|\right|^2, \tag{62}$$

it is seen that the variance of $\hat{\theta}$ can be written in the form

$$\sigma_\theta^2 = E(\hat{\theta} - \theta)^2 = \frac{1}{\rho}\left|\left|\mathrm{g}_\theta - \left(\mathrm{g}_\theta, \frac{\mathrm{g}_\varphi}{||\mathrm{g}_\varphi||}\right)\frac{\mathrm{g}_\varphi}{||\mathrm{g}_\varphi||}\right|\right|^{-2}. \tag{63}$$

The vector appearing in the denominator of (63) is the component of $\mathrm{g}_\theta$ that is perpendicular to $\mathrm{g}_\varphi$. This means that when $\mathrm{g}_\theta$ and $\mathrm{g}_\varphi$ are perpendicular, $\sigma_\theta^2$ is inversely proportional to $||\mathrm{g}_\theta||^2$.

This is a reasonable result because it states that the more sensitive g is to changes in $\theta$, the better will be the accuracy in measuring $\theta$. When $\mathrm{g}_\theta$ and $\mathrm{g}_\varphi$ are not perpendicular, however, the change produced in g by a change in $\theta$ has a component that could equally well have been produced by a change in $\varphi$. It is reasonable to suppose that this component will reduce the measurement accuracy of $\theta$ and that the *useful* sensitivity of g to changes in $\theta$ is given by the vector $\mathrm{g}_\theta$ minus that component of $\mathrm{g}_\theta$ that lies along $\mathrm{g}_\varphi$, i.e., the vector $\mathrm{g}_\theta - (\mathrm{g}_\theta, \mathrm{g}_\varphi/||\mathrm{g}_\varphi||)\mathrm{g}_\varphi/||\mathrm{g}_\varphi||$. This seems to be the reason why the variance $\sigma_\theta^2$ is inversely proportional to the squared modulus of the latter vector and not to $||\mathrm{g}_\theta||^2$ when $\mathrm{g}_\theta$ and $\mathrm{g}_\varphi$ are not perpendicular.

It will now be shown that the large signal-to-noise ratio angle-accuracy formulas just derived for the $m$-beam coherent receiver are valid also for an $m$-beam incoherent receiver. This means that at large signal-to-noise ratios coherent processing offers no advantages over incoherent processing.

The starting point for this analysis is (51), where $\alpha$ and $s_i(t, \alpha)$ are now given by

$$\begin{gathered} \alpha = [\psi_1, \cdots, \psi_m, A, \theta, \varphi] \\ s_i(t, \alpha) = Ae^{i\psi_i}G_i(\theta, \varphi)s(t) \qquad i = 1, \cdots, m. \end{gathered} \tag{64}$$

[7] Two non-negative definite matrices A and B are said to satisfy $\mathrm{A} \leq \mathrm{B}$ if and only if $\mathrm{B} - \mathrm{A}$ is non-negative definite.

Substitution of (64) into (51) yields the result

$$\Lambda = 2N_0 \left[\begin{array}{c|ccc} A_2 \, \|G\|^2 I_{m\times m} & & 0_{m\times 3} & \\ \hline & \|G\|^2 & A(G, G_\theta) & A(G, G_\varphi) \\ 0_{3\times m} & & A^2 \|G_\theta\|^2 & A^2(G_\theta, G_\varphi) \\ & & \text{symmetric} & A^2 \|G_\varphi\|^2 \end{array}\right]^{-1}, \tag{65}$$

where $I_{m\times m}$ denotes the $(m \times m)$-dimensional identity matrix and $0_{m\times n}$ denotes the $(m \times n)$-dimensional zero matrix.

Application of Frobenius' relation, (54), to (65) leads easily to the following formula for the covariance matrix of the estimates for $A$, $\theta$, and $\varphi$,

$$E\{[\hat{A} - A, \hat{\theta} - \theta, \hat{\varphi} - \varphi]'[\hat{A} - A, \hat{\theta} - \theta, \hat{\varphi} - \varphi]\}$$
$$= 2N_0 \begin{bmatrix} \|G\|^2 & A(G, G_\theta) & A(G, G_\varphi) \\ & A^2 \|G\|^2 & A^2(G_\theta, G_\varphi) \\ \text{symmetric} & & A^2 \|G_\varphi\|^2 \end{bmatrix}^{-1}. \tag{66}$$

Another application of Frobenius' relation now leads to the desired covariance matrix for $\hat{\theta}$ and $\hat{\varphi}$,

$$\Lambda_{\theta\varphi} = \frac{1}{\rho} \begin{bmatrix} \|g_\theta\|^2 & (g_\theta, g_\varphi) \\ (g_\theta, g_\varphi) & \|g_\varphi\|^2 \end{bmatrix}^{-1}. \tag{67}$$

This covariance matrix is the same as the covariance matrix that we derived for the coherent receiver, (56).

As an application of the preceding accuracy formulas, consider a four-beam, two-angle monopulse system employing "product beams" arranged on a rectangular grid, i.e.,

$$\left.\begin{aligned} G_1(\theta, \varphi) &= P_1(\theta - \theta_0)P_2(\varphi - \varphi_0) \\ G_2(\theta, \varphi) &= P_1(\theta - \theta_0)P_2(\varphi + \varphi_0) \\ G_3(\theta, \varphi) &= P_1(\theta + \theta_0)P_2(\varphi - \varphi_0) \\ G_4(\theta, \varphi) &= P_1(\theta + \theta_0)P_2(\varphi + \varphi_0) \end{aligned}\right\}, \tag{68}$$

where $P_1$ and $P_2$ are known functions and $2\theta_0$ and $2\varphi_0$ denote the known beam separations in azimuth and elevation, respectively.

Substitution of (68) into (67) yields, after much tedious but straightforward algebra, the following expression for the asymptotic variance of the azimuth estimate,

$$\sigma_\theta^2 = \frac{1}{\rho} \frac{\left[\dfrac{P_1(\theta + \theta_0)}{P_1(\theta - \theta_0)} + \dfrac{P_1(\theta - \theta_0)}{P_1(\theta + \theta_0)}\right]^2}{\left[\dfrac{P_1'(\theta + \theta_0)}{P_1(\theta + \theta_0)} - \dfrac{P_1'(\theta - \theta_0)}{P_1(\theta - \theta_0)}\right]^2}, \tag{69}$$

where the prime denotes differentation and

$$\rho = \frac{A^2}{2N_0} [P_1^2(\theta + \theta_0) + P_1^2(\theta - \theta_0)] \cdot [P_2^2(\varphi + \varphi_0) + P_2^2(\varphi - \varphi_0)]. \tag{70}$$

A similar expression holds for the asymptotic variance of the elevation estimate and it can be shown that the two estimates are asymptotically uncorrelated.

An interesting special case of (69) can be obtained by assuming that the beam shapes are Gaussian with equal widths.

$$P_1(\theta) = \exp(-\theta^2/2\sigma^2), \quad P_2(\varphi) = \exp(-\varphi^2/2\sigma^2). \tag{71}$$

This assumption leads to the expression,

$$\sigma_\theta^2 = \frac{N_0}{2A^2}\left(\frac{\sigma}{\theta_0}\right)^2 \cdot \exp\frac{(\theta^2 + \theta_0^2)}{\sigma^2} \exp\frac{(\varphi^2 + \varphi_0)^2}{\sigma} \frac{\cosh(2\theta\theta_0/\sigma^2)}{\cosh(2\theta\theta_0/\sigma^2)}. \tag{72}$$

The accuracy formulas also are useful for comparing an optimum monopulse system with various suboptimal systems, which may be desirable because of the ease with which they can be implemented or for other reasons. An example of such a suboptimal system is a "conventional" four-beam coherent monopulse system. Such a system starts with four beams of the form

$$\begin{aligned} G_1(\theta, \varphi) &= G(\theta - \theta_0, \varphi - \varphi_0) \\ G_2(\theta, \varphi) &= G(\theta - \theta_0, \varphi + \varphi_0) \\ G_3(\theta, \varphi) &= G(\theta + \theta_0, \varphi - \varphi_0) \\ G_4(\theta, \varphi) &= G(\theta + \theta_0, \varphi + \varphi_0) \end{aligned} \tag{73}$$

and their associated matched filter outputs $y_1$, $y_2$, $y_3$, and $y_4$. An estimate of the azimuth angle $\theta$ is obtained by applying two-beam one-angle maximum-likelihood processing, (30) and (31), to the derived signals $y_1 + y_2$ and $y_3 + y_4$, assuming that the appropriate antenna gain functions are $G_1(\theta, 0) + G_2(\theta, 0)$ and $G_3(\theta, 0) + G_4(\theta, 0)$, respectively. In other words, the "conventional" receiver arrives at an azimuth estimate $\hat{\theta}$ by solving

$$\frac{G_1(\hat{\theta}, 0) + G_2(\hat{\theta}, 0)}{G_3(\hat{\theta}, 0) + G_4(\hat{\theta}, 0)} = \frac{\mathrm{Re}\,[(y_1 + y_2)e^{-i\hat{\psi}}]}{\mathrm{Re}\,[(y_3 + y_4)e^{-i\hat{\psi}}]}, \tag{74}$$

where

$$\hat{\psi} = \tfrac{1}{2} \arg\,[(y_1 + y_2)^2 + (y_3 + y_4)^2]. \tag{75}$$

In actual practice, the sum-and-difference signals $(y_1 + y_2) + (y_3 + y_4)$ and $(y_1 + y_2) - (y_3 + y_4)$ with the associated gains $[G_1(\theta, 0) + G_2(\theta, 0)] + [G_3(\theta, 0) + G_4(\theta, 0)]$ and $[G_1(\theta, 0) + G_2(\theta, 0)] - [G_3(\theta, 0) + G_4(\theta, 0)]$ are used to determine $\hat{\theta}$. This is equivalent to the procedure defined by (74) and (75), as was pointed out earlier.

The elevation estimate $\hat{\varphi}$ is obtained by applying a similar procedure to the derived signals $y_1 + y_4$ and $y_2 + y_3$ using $G_1(0, \varphi) + G_4(0, \varphi)$ and $G_2(0, \varphi) + G_3(0, \varphi)$ as gain functions.

The reasoning behind the procedure just described is based on the fact that the beam shape $G$ and the beam separations $\theta_0$ and $\varphi_0$ usually can be chosen so that $G_1(\theta, \varphi) + G_2(\theta, \varphi)$ and $G_3(\theta, \varphi) + G_4(\theta, \varphi)$ are approximately independent of $\theta$. This being the case, it is natural to attempt to simplify the receiver by assuming the above mentioned gains are exactly independent of $\varphi$ or $\theta$ as the case may be. This assumption leads directly to the estimation procedure described by (74) and (75).

The large signal-to-noise ratio performance of the system decribed by (74) and (75) can be derived by approximating the right-hand side of (74) to first-order noise terms and the left-hand side to first-order terms in $\hat{\theta} - \theta$, where $\theta$ denotes the true azimuth angle. The expansion of the right-hand side uses the true gains, $G_1(\theta, \varphi) + G_2(\theta, \varphi)$ and $G_3(\theta, \varphi) + G_4(\theta, \varphi)$, rather than the assumed gains, $G_1(\theta, 0) + G_2(\theta, 0)$ and $G_3(\theta, 0) + G_4(\theta, 0)$. The results of this procedure are that the aximuth estimate $\hat{\theta}$ has an asymptotic mean given by

$$E(\hat{\theta}) = \theta + \frac{\dfrac{\Gamma_2(\theta)}{\Gamma_1(\theta)} \dfrac{G_1(\theta, \varphi) + G_2(\theta, \varphi)}{G_3(\theta, \varphi) + G_4(\theta, \varphi)} - 1}{\dfrac{\Gamma_1'(\theta)}{\Gamma_1(\theta)} - \dfrac{\Gamma_2'(\theta)}{\Gamma_2(\theta)}}, \tag{76}$$

where the prime denotes differentation, $\theta$ and $\varphi$ are the true azimuth and elevation, and

$$\begin{aligned} \Gamma_1(\theta) &= G_1(\theta, 0) + G_2(\theta, 0) \\ \Gamma_2(\theta) &= G_3(\theta, 0) + G_4(\theta, 0). \end{aligned} \tag{77}$$

The asymptotic variance of $\hat{\theta}$ is given by

$$\sigma_c^2(\hat{\theta}) = \frac{4N_0}{A^2} \frac{\left[\dfrac{\Gamma_2(\theta)}{\Gamma_1(\theta)}\right]^2 \left\{1 + \left[\dfrac{G_1(\theta, \varphi) + G_2(\theta, \varphi)}{G_3(\theta, \varphi) + G_4(\theta, \varphi)}\right]^2\right\}}{\left[\dfrac{\Gamma_1'(\theta)}{\Gamma_1(\theta)} - \dfrac{\Gamma_2'(\theta)}{\Gamma_2(\theta)}\right]^2 [G_3(\theta, \varphi) + G_4(\theta, \varphi)]^2}. \tag{78}$$

Specialization of these formulas to the case of the product beams defined by (68) yields the result,

$$E(\hat{\theta}) = \theta$$

$$\sigma_c^2(\hat{\theta}) = \frac{2}{\rho'} \frac{\left[\dfrac{P_1(\theta + \theta_0)}{P_1(\theta - \theta_0)} + \dfrac{P_1(\theta - \theta_0)}{P_1(\theta + \theta_0)}\right]^2}{\left[\dfrac{P_1'(\theta + \theta_0)}{P_1(\theta + \theta_0)} - \dfrac{P_1'(\theta - \theta_0)}{P_1(\theta - \theta_0)}\right]^2} \tag{79}$$

where

$$\rho' = \frac{A^2}{2N_0} [P_1^2(\theta + \theta_0) + P_1^2(\theta - \theta_0)] \cdot [P_2(\varphi + \varphi_0) + P_2(\varphi - \varphi_0)]^2. \tag{80}$$

Comparison of (79) and (69) shows that the ratio of the estimation variance of the optimum system to that of the "conventional" system is given by

$$\frac{\sigma^2(\hat{\theta})}{\sigma_c^2(\hat{\theta})} = \frac{1}{2} \frac{\rho'}{\rho} = \frac{1}{2} \left[1 + \frac{P_2(\varphi + \varphi_0)}{P_2(\varphi - \varphi_0)}\right]^2 \left[1 + \frac{P_2^2(\varphi + \varphi_0)}{P_2^2(\varphi - \varphi_0)}\right]^{-1}. \tag{81}$$

It is easy to bound the right-hand side of (81) above and below with the result[8]

$$\tfrac{1}{2} \leq \sigma^2(\hat{\theta})/\sigma_c^2(\hat{\theta}) \leq 1, \tag{82}$$

which states that the azimuth-estimation variance of the conventional system can be no more than twice the variance of the optimum system. Stated in somewhat different terms, this means that the asymptotic azimuth-estimation accuracy of a "conventional" monopulse system can be made equal to that of an optimum system by increasing the transmitted signal power by no more than 3 dB.

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[8] The lower bound is based on the assumption that $P_2(\varphi + \varphi_0)/P_2(\varphi - \varphi_0) \geq 0$. This condition will be met, at least for angular directions of interest, in any well designed monopulse system. Violation of this condition implies that signals are being added in phase opposition when the beams $G_1 + G_2$ and $G_3 + G_4$ are formed. This is a situation which surely will be avoided in the design of any practical monopulse system.