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The Input Impedances of Slit Antennas.

By

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§ 1. Introduction

Various works on the calculation of the radiation impedances of slit antennas have been carried out by several investigators.[1][2] But they used only the so-called "M. M. F. method" which corresponds to the so-called "E. M. F. methed" generally used in the theoretical treatment of ordinary linear antennas.

The "E. M. F. method" is not essentially quite accurate because the process is based on the assumption of unknown current distribution. Whereas, Hallén's method,[3] under which the impedances are calculated directly from the ratios of the applied voltages to the determined input currents by using Maxwell's fundamental equations and boundary conditions, seems to be the more accurate and important. Therefore in order to solve the problems of slit antennas it is also better not to use the "M. M. F. method" which needs assuming magnetic current distributions.

In this paper we considered at first general relations existing between the electromagnetic field due to holes in a perfectly conducting infinite plane and the field due to the dual plates which have the same shapes as the holes by introducing magnetic perfect conductor. Next by using these relations we calculated directly the input impedance of a slit antenna from the voltage and current relation without using the idea of radiating power used as in "M. M. F. method". And finally general relations between the input impedances of slit antennas and those of plate antennas which are dual with the slits were made clear.

§ 2. Duality between two electromagnetic fields due to interchanging electric and magnetic current sources with each other.

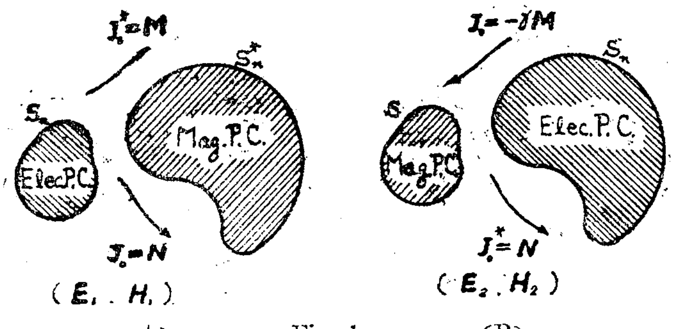
Suppose that any number of electric perfect conductors and magnetic perfect conductors be placed in a homogeneous isotropic medium whose electromagnetic constants are respectively ε, σ and μ. Let S_1, S_2, ... be the surfaces of these electric perfect conductors, and S_1^*, S_2^*,..... be those of magnetic perfect conductors, the field equations and the conditions for continuity of electric and magnetic charges can be written as follows:

$$\left.\begin{aligned}&\nabla\times E+j\omega\mu H=-J_0^*,\\&\nabla\times H-(j\omega\varepsilon+\sigma)E=J_0,\\&\nabla\cdot H=\frac{\rho^*}{\mu},\\&\nabla\cdot E=\frac{\rho}{\varepsilon},\\&\nabla\cdot J_0^*+j\omega\rho^*=0,\\&\nabla\cdot J_0+j\omega\left(1+\frac{\sigma}{j\omega\varepsilon}\right)\rho=0,\end{aligned}\right\}\quad(1)$$

where J_0, J_0^* are electric and magnetic current sources respectively, and we assume that the distribution of these sources is given independently of other quantities and that their origins are outside of our considerations. Also ε is the dielectric constant, μ the permeability, σ the conductivity, ρ, ρ^* the electric and magnetic charge densities respectively, and M. K. S. rational unit system is used. Unless otherwise noticed, we use the same notations as in Stratton's "Electromagnetic Theory"(4) except the time factor which are substituted by $e^{j\omega t}$.

Furthermore we must take following relations as boundary conditions.

$$\left.\begin{aligned}&E\times n=H\cdot n=0, \text{ (over } S_1, S_2,\cdots\cdots),\\&H\times n=E\cdot n=0, \text{ (over } S_1^*, S_2^*,\cdots\cdots)\end{aligned}\right\}\quad(2)$$



(A) Fig. 1 (B)
Dual electromagnetic fields due to interchanged source distributions and boundary conditions.

Let $E_1=F$, $H_1=G$ in (A) of Fig. 1 be the electromagnetic field vectors in the case where electric and magnetic current distributions are given by the equations

$$\left.\begin{aligned}&J_0=N,\ J_0^*=M,\\&\rho=n,\ \rho^*=m\end{aligned}\right\}\quad(3)$$

where

$$\left.\begin{aligned}&\nabla\cdot M+j\omega m=0,\\&\nabla\cdot N+j\omega\left(1+\frac{\sigma}{j\omega\varepsilon}\right)n=0,\end{aligned}\right\}\quad(4)$$

Then F, G satisfy the field equations

$$\left.\begin{aligned}&\nabla\times F+j\omega\mu G=-M,\\&\nabla\times G-(j\omega\varepsilon+\sigma)F=N,\\&\nabla\cdot G=\frac{m}{\mu},\\&\nabla\cdot F=\frac{n}{\varepsilon}\end{aligned}\right\}\quad(5)$$

and the boundary conditions become

$$\left.\begin{aligned}&F\times n=G\cdot n=0,\ (\text{over } S_1,\ S_2,\cdots\cdots),\\&G\times n=F\cdot n=0,\ (\text{over } S_1^*,\ S_2^*,\cdots\cdots)\end{aligned}\right\}\quad(6)$$

Let us put

$$\gamma=\frac{\varepsilon+\dfrac{\sigma}{j\omega}}{\mu}\quad(7)$$

Interchanging the order of arrangement of the equations (5) and modifying them, it gives

$$\left.\begin{aligned}&\nabla\times(-G)+j\omega\mu(\gamma F)=-N,\\&\nabla\times(\gamma F)-(j\omega\varepsilon+\sigma)(-G)=(-\gamma M),\\&\nabla\cdot(\gamma F)=\frac{\left(1+\dfrac{\sigma}{j\omega\varepsilon}\right)n}{\mu},\\&\nabla\cdot(-G)=\frac{(-\gamma m)}{\varepsilon\left(1+\dfrac{\sigma}{j\omega\varepsilon}\right)}\end{aligned}\right\}\quad(8)$$

Similarly Eqs. (4) become

$$\left.\begin{aligned}&\nabla\cdot N+j\omega\left(1+\frac{\sigma}{j\omega\varepsilon}\right)n=0,\\&\nabla\cdot(-\gamma M)+j\omega\left(1+\frac{\sigma}{j\omega\varepsilon}\right)\frac{(-\gamma m)}{1+\dfrac{\sigma}{j\omega\varepsilon}}=0\end{aligned}\right\}\quad(9)$$

Again, let E_2, H_2 in (B) of Fig. 1 be the electromagnetic fields due to the new source distributions given in the following equation (10) in the same space as in the preceding case except for the difference of boundary conditions which are caused by converting electric perfect conductors into magnetic perfect conductors and vice versa.

$$\left.\begin{aligned} J_0&=-\gamma M, \qquad J_0^*=N, \\ \rho&=\frac{-\gamma m}{1+\dfrac{\sigma}{j\omega\varepsilon}}, \quad \rho^*=\left(1+\frac{\sigma}{j\omega\varepsilon}\right)n \end{aligned}\right\} \quad (10),$$

If we compare Eqs. (8) (9) (10) with Eqs. (1) (4) (3) we can easily find the relations

$$\left.\begin{aligned} E_2&=-G=-H_1, \\ H_2&=\gamma F=\gamma E_1 \end{aligned}\right\} \quad (11)$$

Because E_2 and H_2 satisfy not only the field equations, but also new boundary conditions in virtue of the relations given in Eqs. (6), (8).

Thus it can be verified that there exists the relation of duality between two electromagnetic fields due to the distributions interchanging electric and magnetic current sources.

§ 3. E-symmetry and H-symmetry.[5]

We will discuss about symmetrical and anti-symmetrical components of an electromagnetic field with respect to the plane $x=0$ in rectangular coordinates x, y, z.

For simplicity, let us denote a three-dimensional function $E(x, y, z)$ by $E(x)$. Then we can separate $E(x)$ as follows:

$$\begin{aligned} E(x)=&\frac{1}{2}\Big[\, i\{E_x(x)-E_x(-x)\}+j\{E_y(x)+E_y(-x)\}+k\{E_z(x)+E_z(-x)\}\Big] \\ &+\frac{1}{2}\Big[\, i\{E_x(x)+E_x(-x)\}+j\{E_y(x)-E_y(-x)\}+k\{E_z(x)-E_z(-x)\}\Big] \quad (12) \end{aligned}$$

The first bracketed term in the right-hand side of this equation is symmetrical and the second is antisymmetrical. Representing these two components of $E(x)$ by the notations, $E_s(x)$, $E_a(x)$ respectively, Eq. (12) becomes

$$E(x)=E_s(x)+E_a(x) \quad (13)$$

Similarly $H(x)$, $J(x)$ etc. can be separated into two components, and hence Eqs. (1) can be transformed to

$$\left.\begin{aligned} &\{\nabla\times E_a+j\omega\mu H_s+J_{0s}^*\}+\{\nabla\times E_s+j\omega\mu H_a+J_{0a}^*\}=0, \\ &\{\nabla\times H_s-(j\omega\varepsilon+\sigma)E_a-J_{0a}\}+\{\nabla\times H_a-(j\omega\varepsilon+\sigma)E_s-J_{0s}\}=0 \end{aligned}\right\} \quad (14)$$

Let $\nabla\times E_a(x)$ be the value of $\nabla\times E_a$ for x, then from its definition we find

$$\left.\begin{aligned}
\left\{\nabla\times E_a(x)\right\}_x &= \frac{\partial E_{az}(x)}{\partial y}-\frac{\partial E_{ay}(x)}{\partial Z}=-\left\{\frac{\partial E_{az}(-x)}{\partial y}-\frac{\partial E_{ay}(-x)}{\partial Z}\right\}\\
&=-\left\{\nabla\times E_a(-x)\right\}_x,\\
\left\{\nabla\times E_a(x)\right\}_y &= \frac{\partial E_{ax}(x)}{\partial Z}-\frac{\partial E_{az}(x)}{\partial x}=\frac{\partial E_{ax}(-x)}{\partial Z}-\frac{\partial E_{az}(-x)}{\partial(-x)}\\
&=\left\{\nabla\times E(-x)\right\}_y,\\
\text{Likwise}\quad&\\
\left\{\nabla\times E_a(x)\right\}_z &= \left\{\nabla\times E_a(-x)\right\}_z
\end{aligned}\right\}\quad(15)$$

These relations show that the vector $\nabla\times E_a$ is symmetrical with respect to the plane $x=0$. In the same manner we can show that $\nabla\times H_a$ is symmetrical, and that $\nabla\times E_s$, $\nabla\times H_s$ are anti-symmetrical.

Furthermore, if the structure of space is symmetrical, Eq. (14) are independent of the sign of x. Take one-half of the sum and the difference of two sets of equations, one of which arises from changing the sign of x in the set of equations (14) and the other is the set of equations (14) itself. Then, in virtue of symmetrical or anti-symmetrical property of each term, we obtain

$$\left.\begin{aligned}
&\nabla\times E_a+j\omega\mu H_s=-J_{0s}{}^*,\\
&\nabla\times H_s-(j\omega\varepsilon+\sigma)E_a=J_{0a}
\end{aligned}\right\}(16)$$

$$\left.\begin{aligned}
&\nabla\times E_s+j\omega\mu H_a=-J_{0a}{}^*,\\
&\nabla\times H_a-(j\omega\varepsilon+\sigma)E_s=J_{0s}
\end{aligned}\right\}(17)$$

According to Eq. (16) the electromagnetic field E_a and H_s due to the sources $J_{0s}{}^*$, J_{0a} are magnetically symmetrical, and hence it can be called a field of H-symmetry. Likewise, from Eq. (17) we know that the electromagnetic field E_s and H_a due to $J_{0a}{}^*$, J_{0s} are electrically symmetrical or of E-symmetry.

As the electromagnetic field must be continuous in a homogenious medium, from the definition of E-symmetry we find

$$\begin{aligned}
E_{sx}(+0)&=-E_{sx}(-0),\\
H_{ay}(+0)&=-H_{ay}(-0),\\
H_{az}(+0)&=-H_{az}(-0).
\end{aligned}$$

Therfore

$$E_{sx}(0)=H_{ay}(0)=H_{az}(0)=0 \qquad (18)$$

This shows that the field satisfies a boundary condition of a perfect magnetic conductor at every point on the plane $x=0$ in a homogeneous medium. Consequently, even if we insert an arbitrarily shaped and sized plane sheet of perfect magnetic conductor in the medium at $x=0$, a field of E-symmetry dose not change its intensity at any point. Likewise similar relation holds for a plane sheet of perfect electric conductor and a field of H-symmetry.

§ 4. A plate and a hole which are mutually dual.[8]

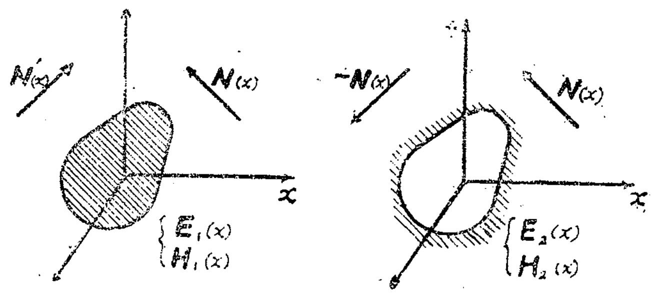


Fig. 2

(A) Symmetrical electric current source and a plate of perfect conductor.

(B) Anti-symmetrical magnetic current and a hole on an infinitely large plane sheet of perfect conductor

If a plate of perfect electric conductor is placed on the plane S at $x=0$, in an isotropic homogeneous medium, and a symmetrical distribution of electric current J_{0s} is given by the following equations

$$J_{0s}(x)=N(x),\ (x>0),\qquad J_{0s}(x)=N'(x),\ (x<0) \tag{19}$$

where N, N' satisfy the symmetrical relation

$$N_x(x)=-N'_x(-x),\ N_y(x)=N_y'(-x),\ N_z(x)=N_z'(-x) \tag{20}$$

then, in virtue of Eq. (17) the induced field E, H is E-symmetrical.

Therefore we can put

$$\left.\begin{array}{l} E_1(x)=F(x) \\ H_1(x)=G(x) \end{array}\right\}(x>0),\quad \left.\begin{array}{l} E_1(x)=F'(x) \\ H_1(x)=G'(x) \end{array}\right\}(x<0) \tag{21}$$

where F, F', G, G', are vector functions whose components are related by the following equations

$$\left.\begin{array}{l} F_x(x)=-F_x'(-x) \\ F_y(x)=F_y'(-x) \\ F_z(x)=F_z'(-x) \end{array}\right\},\quad \left.\begin{array}{l} G_x(x)=G_x'(-x) \\ G_y(x)=-G_y'(-x) \\ G_z(x)=-G_z'(-x) \end{array}\right\} \tag{22}$$

Now, we cover the whole plane $x=0$ except S, say $\overline{S}$, with a plate of magnetic perfect conductor. Then by the result of preceding paragraph we find that the space is separated into two parts, each of which has independent boundary, without changing the field intensity at any point.

Next we alter the nature of perfect conductors over $\overline{S}$ and S, from electric to magnetic, and from magnetic to electric, respectively. Moreover we replace the symmetrical electric current by an anti-symmetrical distribution of impressed magnetic current

$$J_{0a}^*(x)=N(x),\ (x>0),\quad J_{0a}^*(x)=-N'(x),\ (x<0) \tag{23}$$

Then, the induced field E_2, H_2 are given by the following equations, because the result of § 2 can be applied for two half-spaces,, $x>0$ and $x<0$, separately.

$$\left.\begin{array}{l}E_2(x)=-G(x)\\ H_2(x)=\gamma F(x)\end{array}\right\}(x>0),\qquad \left.\begin{array}{l}E_2(x)=G'(x)\\ H_2(x)=-\gamma F'(x)\end{array}\right\}(x<0) \tag{24}$$

Therefore according to the relations (22) we find that this field is E-symmetrical. Thus by the result of § 3 the magnetic perfect conductor over the plane S can be taken away without disturbing the distribution of field intensity.

Consequently we know that the problem of perfect electric conductor plate over S under a symmetrical distribution of impressed electric current $J_{0s}(x)$ as shown in Fig. 2 (A), and the problem of holed sheet conductor over $\overline{S}$ under an anti-symmetrical impressed magnetic current $J_{0a}^*(x)$ as shown in Fig. 2 (B), are mutually reducible from one to another. That is, the solutions of such problems have the relations

$$\left.\begin{array}{l}E_2(x)=-H_1(x)\\ H_2(x)=\gamma E_1(x)\end{array}\right\}(x>0),\qquad \left.\begin{array}{l}E_2(x)=H_1(x)\\ H_2(x)=-\gamma E_1(x)\end{array}\right\}(x<0) \tag{25}$$

§ 5. The input impedance of slit antenna.

Now let us consider an arbitrarily shaped slit antenna of Fig. (3), whose feeding points a and b are connected to an electric source. This electric source can be interpreted as a magnetic source which is given around a strip of conductor connecting a and b as illustrated in Fig. 4. It is natural to consider that the magnctic source is equally distributed on both sides of the strip, but it must be noted that the direction of its action on the backside is reverse to that of the front side. Accordingly this magnetic source can be interpreted as an anti-symmetrical distribution of impressed magnetic current given by Eqs. (23).

Let us reduce next this problem of slit antenna to a problem of a dual plate antenna. Then, by the result of § 4 the relations (25) exist between the field E_1, H_1 for the plate antenna of Fig. 3, and the field E_2, H_2 for the slit antenna of Fig. 5. Again let V, I be the voltage and current of electric source connected to the feeding points a, b, in Fig. 3, and V', I', be that of c, d in

Fig, 5. Then, as these electromagnetic fields are E-symmetrical, it gives

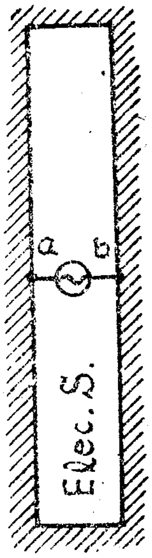
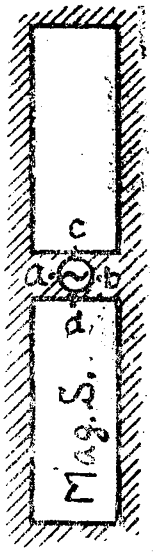


Fig. 3
A slit antenna fed by an electric source.



Fig, 4
A slit antenna fed by a magnetic source.

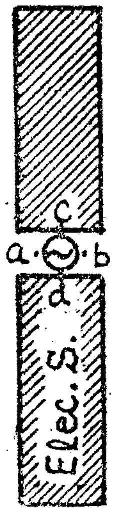


Fig. 5
A plate antenna fed by an electric source.

$$\left.\begin{aligned} V&=-\int_{(+)a}^{b} E_2\cdot dl, \\ I&=\oint_{cdc} H_2\cdot dl=\int_{(+)c}^{d} H_2\cdot dl+\int_{(-)d}^{c} H_2\cdot dl=2\int_{(+)c}^{d} H_2\cdot dl \end{aligned}\right\}\quad(26)$$

$$\left.\begin{aligned} V'&=-\int_{(+)d}^{c} E_1\cdot dl, \\ I'&=\oint_{aba} H_1\cdot dl=\int_{(+)a}^{b} H_1\cdot dl+\int_{(-)b}^{a} H_1\cdot dl=2\int_{(+)a}^{b} H_1\cdot dl \end{aligned}\right\}\quad(27)$$

where the path of these line integrals are straight line on the surface of strip in which the source is included, the term $\oint$ must be integrated from the front side, and $(+)$ or $(-)$ denotes the integration on the front or back side of the surface respectively.

From Eqs. (25), (26) and (27) we find

$$\left.\begin{aligned} V&=\int_{(+)a}^{b} H_1\cdot dl=\frac{1}{2}I', \\ I&=2\int_{c}^{d} E_1\cdot dl=2\gamma V' \end{aligned}\right\}\quad(28)$$

Let Z be the input impedance of slit antenna at a,b and Z' be that of plate antenna at c, d, then by the relations of Eq. (28) we have

$$Z=\frac{V}{I}=\frac{1}{4\gamma}\cdot\frac{I'}{V'}=\frac{1}{4\gamma Z'} \tag{29}$$

In the free space, on the other hand, Eq. (7) reduces to

$$\frac{1}{\gamma}=(120\pi)^2 \tag{30}$$

Introducing this equation into Eq. (29), we obtain

$$ZZ'=(60\pi)^2 \tag{31}$$

The relations (28), (29) and (31) are derived without any assumption such as distribution of magnetic current, and so the shape of slit is quite arbitrary. Accordingly the input impedance of one-half of infinitely large plane sheet of perfect conductor which is formed by a revolutionarily or axially symmetrical figures and fed at a, b, as illustrated in Fig. 6, is independent of frequency and its shape, and is given always by the expression

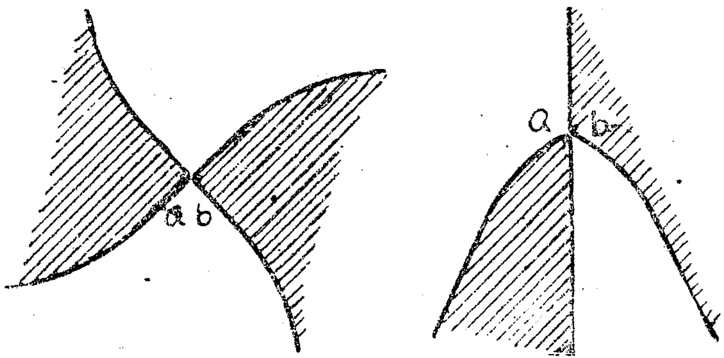


Fig. 6 One-half of infinitely large plane sheet of perfect conductor, which has constant input impedance.

$$Z=\frac{1}{\sqrt{4\gamma}} \tag{32}$$

where the sign of square root must be taken such that the real part of Z is positive. For example, in the free space Eq. (32) reduces to

$$Z=60\pi \text{ ohm.}$$

Now, we consider a rectangular slit antenna fed at the middle of it, whose length and breadth are $2l$ and a, respectively.

As the equivalent radius of a rectangular narrow plate antenna which has the breadth a is $a/4$, well known Hallén's result can be applied to the case $\sigma=0$ as follows:

$$Z'=\frac{1}{j2\pi}\sqrt{\frac{\mu}{\varepsilon}}\frac{\Omega\cos kl+\alpha_1+\frac{\alpha_2}{\Omega}}{\sin kl+\frac{\beta_1}{\Omega}+\frac{\beta_2}{\Omega^2}} \tag{33}$$

where

$$\Omega=2\log\frac{8l}{a} \tag{34}$$

α_1, α_2, β_1, β_2 are the functions of l. (see appendix) By using Eq. (29) we obtain

$$Z=\frac{j\pi}{2}\sqrt{\frac{\mu}{\varepsilon}}\frac{\sin kl+\frac{\beta_1}{\Omega}+\frac{\beta_2}{\Omega^2}}{\Omega\cos kl+\alpha_1+\frac{\alpha_2}{\Omega}}\ \text{Ohm} \tag{35}$$

If the length $2l$ is nearly equal to a half wave length $\lambda/2$, in the right-hand

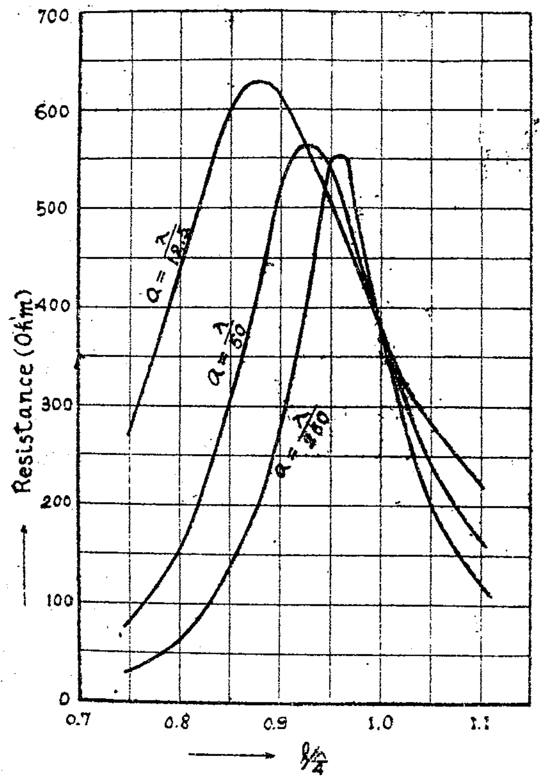


Fig. 7 The input resistance of a rectangular slit antenna.

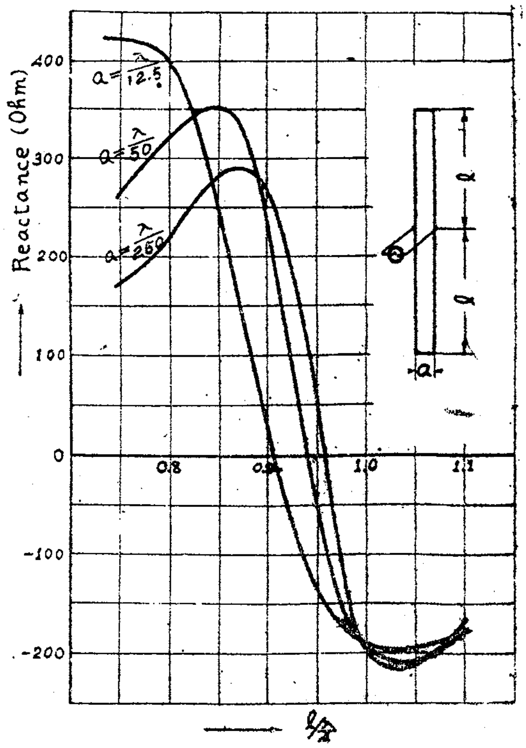


Fig. 8 The input reactance of a rectangular slit antenna.

side of this equation the influence of β_2/Ω^2 on the numerator may be smaller than that of α_2/Ω on the denominator, because $\cos kl \fallingdotseq 0$, while $\sin kl \fallingdotseq 1$. Hence we neglect the term β_2/Ω^2. Thus, in free space we obtain the following approximate equation

$$Z \fallingdotseq j60\pi^2 \frac{\sin kl + \dfrac{\beta_1}{\Omega}}{\Omega\cos kl + \alpha_1 + \dfrac{\alpha_2}{\Omega}} \text{ Ohm}, \left(l \fallingdotseq \frac{\lambda}{2}\right) \tag{36}$$

In Fig. 7 and Fig. 8 real and imaginary part of Z, i.e. resistance R and reactance X, are plotted against l.

When the length of slit is nearly equal to one wave length, a similar approximation can be done.

§ 6. Slit antenna system.

Let us investigate on the relation which exists between the general equations at the feeding points of a slit antennas and those of dual plate antennas.

Let V_s, I_s, V_s', I_s' be respectively the impressed voltages and the feeding currents of mutually dual slit and plate antenna systems. Then the general relasions of these quantities may be written in the form

$$V_r = \sum_{s=1}^{n} Z_{rs} I_s \tag{37}$$

$$V_r' = \sum_{s=1}^{n} Z_{rs}' I_s' \tag{38}$$

It goes without saying that these two equations are not derived from the assumption of current or voltate distribution as made by other investigators, but derived from Maxwell's equations and boundary conditions only.

On the other hand, as the system V, I and V', I' are mutually independent, such a feeding that satisfies the similar relations to Eq. (28) can exist: i. e.

$$\left.\begin{aligned} V_r &= \frac{1}{2} I_r', \\ I_r &= 2\gamma V_r' \end{aligned}\right\} \tag{39}$$

Eliminating V_r', I_r' from Eqs. (38), (39), we obtain

$$\frac{1}{4\gamma} I_r = \sum_{s=1}^{n} Z_{rs}' V_s \tag{40}$$

or solving for V_s, leads to

$$V_r = \frac{1}{4\gamma} \sum_{s=1}^{n}{}' Y_{rs}' I_s \tag{41}$$

where the matrix of Y_{rs}', or $[Y']$ is inverse matrix of $[Z']$.

Now we compare Eq. (41) with Eq. (37). As every I_s is independent and can take quite arbitrary value, there must be the condition

$$Z_{rs} = \frac{1}{4\gamma} Y_{rs}' \tag{42}$$

or expressed by matrix

$$[Z] = \frac{1}{4\gamma} [Z']^{-1} \tag{43}$$

The result may be said as follows. The input impedances of a slit antenna system Z_{rs} are determined from those of dual plate antennas by means of Eq. (43) and also there holds always the general equation (37).

§ 7. Conclusion.

So far as the authors are aware Eqs. (29) (43) are derived always from the assumption of distribution of magnetic current on the slit antennas by means of the idea of radiated electric power(1)(2)(7). The result of this study, however, shows that these equations can be derived directly from the relation of voltage and current without any such assumption, and hence it is clear that these results can be applied to arbitrarily shaped slit antennas.

For the input impadances of slit antennas, the expression formerly obtaind by other authors is the one which corresponds only to that of Labus(8) for ordinary half wave length linear antenna. But in this paper the expression which corresponds to Hallén's equation is derived. Accordingly Eq. (35) is directly applicable for a slit which has arbitrary length, while the former expression can not be applied for the slit, the length of which is nearly equal to one wave length. If the length of slit is nearly equal to half wave length, there is no great difference in their results, but the authors convince that the idea taken in the above method is more progressive one than so-called "M. M. F. method" which is based upon an assumption of distribution of magnetic current.

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Appendix.

According to the result of Hallén α_1, β_1, and α_2 for perfect conductor are given as follows:

$$\alpha_1=\frac{1}{2}\Big[\cos kl\ \{C(4kl)-2C(2kl)\}-\sin kl S(4kl)\Big]$$
$$+\frac{j}{2}\Big[\cos kl\ \{S(4kl)-2S(2kl)\}+\sin kl C(4kl)\Big],$$

$$\beta_1=\frac{1}{2}\Big[\cos kl\ \{4S(2kl)-S(4kl)\}+\sin kl\ \{2C(2kl)-C(4kl)+4\log 2\}\Big]$$
$$+\frac{j}{2}\Big[\cos kl\ \{C(4kl)-4C(2kl)\}+\sin kl\ \{2S(2kl)-S(4kl)\}\Big]$$

$$\alpha_2=-\int_{-l}^{l}\frac{\{F_1(\xi)-F_1(l)\}\ e^{-jk(l-\xi)}}{l-\xi}d\xi,$$

where

$$C(z)=\gamma+\log z-Ci(z),$$

$$S(z)=Si(z),\qquad \gamma=0.5772156\cdots\cdots,$$

$$F_1(x)=-(\cos kx-\cos kl)\log\frac{l^2-x^2}{l^2}$$

$$+\frac{1}{2}\cos kx\left[C\{2k(l+x)\}+C\{2k(l-x)\}+jS\{2k(l+x)\}+jS\{2k(l-x)\}\right]$$

$$-\frac{1}{2}\sin kx\left[S\{2k(l+x)\}-S\{2k(l-x)\}-jC\{2k(l+x)\}+jC\{2k(l-x)\}\right]$$

$$-\cos kl\left[C\{k(l+x)\}+C\{k(l-x)\}+jS\{k(l+x)\}+jS\{k(l-x)\}\right],$$

(β_2 is omitted.)