Reprints from the Early Days of Information Sciences

Reminiscences of the Early Work in Walsh Functions

Interviews with

Franz Pichler
William R. Wade
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Edited by Radomir S. Stanković and Jaakko T. Astola

Tampere International Center for Signal Processing
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Preface

Walsh functions were introduced in 1923 by Joseph Leonard Walsh as a particular orthogonal basis in the space of square-integrable functions on the unit interval [0,1). This set of 1,-1 valued functions can be alternatively viewed as group characters of the infinite dyadic group, as it was shown in 1949 by Fine.

A considerable interest in Walsh functions and their discrete counterparts, was raised in early seventies, as can be seen from the organization of five international conferences on Walsh functions and their applications from 1970 to 1974 in Washington, DC sponsored by the US Navy. In 1971 and 1973 were organized two Symposia on Walsh and Other Non-sinusoidal Functions Applications, at Hatfield Polytechnic, England.

These high research interests and related activities led to the rise of *Spectral techniques* as a mathematical discipline in the area of *abstract harmonic analysis* devoted to applications in engineering, primarily electrical and computer engineering. In this context, the discrete Walsh functions are viewed as group characters of finite dyadic groups consisting of the set of binary *n*-tuples enriched with the componentwise addition modulo 2.

The interest in mathematical and engineering applications motivated transferring of results from the classical Fourier analysis on the real line to the Walsh-Fourier analysis on both infinite and finite dyadic groups, as well as derivation of new results specific for the analysis in terms of Walsh and discrete Walsh functions.
This booklet presents short interviews of three pioneering researchers in Walsh functions, Professor Franz Pichler of Johannes Kepler University, Linz, Austria, Professor William R. Wade of the Department of Mathematics, University of

Tennessee, Knoxville, TN, USA, Professor Ferenz Schipp of Eötvös Lorand University, Budapest, Hungary. They were asked a simple question *How did you start to work in Walsh functions?*

We talked with Prof. Schipp and Prof. Wade during the *Workshop on Dyadic Analysis and Related Areas with Applications* held on June 7 — 10, 2009 in Dobogókő, Hungary. The Workshop was Dedicated to Prof. Ferenc Schipp on the occasion of his 70th birthday and to Prof. Peter Simon also from Eötvös Lorand University on the occasion of his 60th birthday.

The interview with Prof. Pichler was conducted on June 12, 2009 in Vienna.

Radomir S. Stanković, Jaakko T. Astola

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Franz Pichler
Professor Emeritus
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It is so that the inventor of the Walsh functions for Innsbruck was Roman Liedl, he is still there a Professor, maybe he retired already, a mathematician, and he invented as many other researchers also on his own the Walsh functions. Later he found out that the concept exist already, but Liedl already saw also the group relations, group theoretical relations and topological group relations. Then research started and I think that in Innsbruck about twenty PhD theses on Walsh functions were made. Many theses were defended. For example, Peter Weiß, he is still at Linz, was one of the first, and they were mainly devoted to generalized Walsh functions. At that time we looked at the work of Lèvy, and Rice and Selfdrige and others. Selfridge, these were names that passed, and also Vilenkin, the Russian important Walsh function researcher.

Roman Rudolf Liedl, Professor of Mathematics, Institute of Mathematics University of Innsbruck, Austria

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Liedl, R., "Über eine spezielle Klasse von stark multiplikativ orthogonalen Funktionensystemen", *Monatshefte für Mathematik,* 68,1964,130-137.
Liedl, R., Eine algebraische Herleitung und eine Verallgemeinerung des Satzes von Fine über Gruppen von orthonormalen Funktionen und eine Beschreibung der vielfait-erhaltenden Transformationen des Intervalles [0,1] audsich selbst*, Habilitationsschrift, Universitäat,* Innsbruck

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Then Harmuth discovered that there was a Walsh researcher in Innsbruck and he contacted the Innsbruck people. He finally wanted to develop some theoretical framework for his meander functions, which were essentially what have been called *cal* and *sal* functions.

Since I was already starting a PhD work, and they knew, the mathematicians there, Roman Liedl knew that I had a communication background, I was the right one to get interested in that, and so it started, and it was interesting. But of course, at that time we had no overview, for example, I had to discover also the concept of the dyadic filter and this is dyadic convolution, I did not know that before. So this is all separate, you are a student, you do not know, and so it started.

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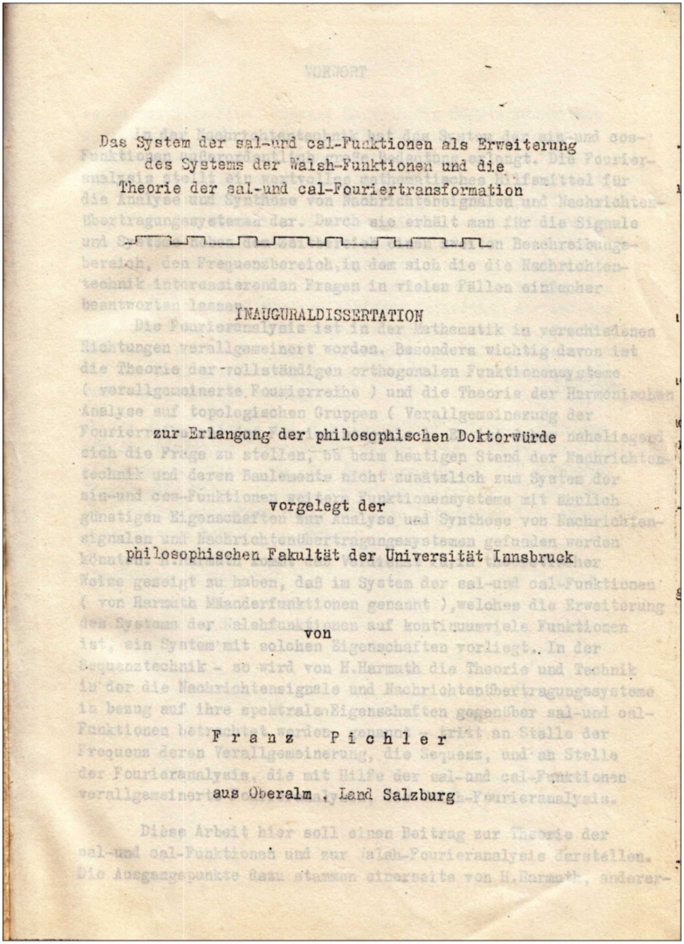

Das System der sal-und cal-Funktionen als Erweiterung
des Systems der Walsh-Funktionen und die
Theorie der sal-und cal-Fouriertransformation

INAUGURALDISSERTATION

zur Erlangung der philosophischen Doktorwürde

vorgelegt der

philosophischen Fakultät der Universität Innsbruck

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VORWORT

In der Nachrichtentechnik hat das System der sin-und cos-Funktionen außerordentlich große Bedeutung erlangt. Die Fourieranalysis stellt ein wertvolles mathematisches Hilfsmittel für die Analyse und Synthese von Nachrichtensignalen und Nachrichtenübertragungssystemen dar. Durch sie erhält man für die Signale und Systeme neben dem Zeitbereich einen zweiten Beschreibungsbereich, den Frequenzbereich,in dem sich die die Nachrichtentechnik interessierenden Fragen in vielen Fällen einfacher beantworten lassen.

Die Fourieranalysis ist in der Mathematik in verschiedenen Richtungen verallgemeinert worden. Besonders wichtig davon ist die Theorie der vollständigen orthogonalen Funktionensysteme (verallgemeinerte Fourierreihe) und die Theorie der Harmonischen Analyse auf topologischen Gruppen (Verallgemeinerung der Fourierreihe und des Fourierintegrals). Es ist daher naheliegend sich die Frage zu stellen, ob beim heutigen Stand der Nachrichtentechnik und deren Baulemente nicht zusätzlich zum System der sin-und cos-Funktionen weitere Funktionensysteme mit ähnlich günstigen Eigenschaften zur Analyse und Synthese von Nachrichtensignalen und Nachrichtenübertragungssystemen gefunden werden könnten. H.Harmuth kommt das Verdienst zu,in theoretischer Weise gezeigt zu haben, daß im System der sal-und cal-Funktionen (von Harmuth Mäanderfunktionen genannt),welches die Erweiterung des Systems der Walshfunktionen auf kontinuumviele Funktionen ist, ein System mit solchen Eigenschaften vorliegt. In der Sequenztechnik - so wird von H.Harmuth die Theorie und Technik in der die Nachrichtensignale und Nachrichtenübertragungssysteme in bezug auf ihre spektralenEigenschaften gegenüber sal-und cal-Funktionen betrachtet werden, genannt - tritt an Stelle der Frequenz deren Verallgemeinerung, die Sequenz, und an Stelle der Fourieranalysis, die mit Hilfe der sal-und cal-Funktionen verallgemeinerte Fourieranalysis, die Walsh-Fourieranalysis.

Diese Arbeit hier soll einen Beitrag zur Theorie der sal-und cal-Funktionen und zur Walsh-Fourieranalysis darstellen. Die Ausgangspunkte dazu stammen einerseits von H.Harmuth, anderer-

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seits von N.J. Fine. Während Harmuth die Mäanderfunktionen $A_{s\mu}(\theta)$ und $A_{c\mu}(\theta)$ mit Hilfe einer Differenzengleichung gewinnt und Fine die " generalized Walsh-functions " $\psi_y(x)$ und $\psi_y^*(x)$, die mit den sal-und cal-Funktionen stark verwandt sind, mit Hilfe der Charaktere einer topologischen Gruppe definiert, gehen wir bei der Definition der sal-und cal-Funktionen in davon verschiedener Weise vor. Die sal-und cal-Funktionen werden bei uns mit Hilfe der stetigen linearen Funktionen eines topologischen Vektorraumes definiert. Wir hoffen daß damit eine anschauliche und einfache Definition der sal-und cal-Funktionen erreicht wurde. Es sei noch erwähnt, daß die Bildung " sal" und " cal " einerseits auf Walsh, andererseits auf die Analogie der sal-und cal-Funktionen zu den sin-und cos-Funktionen hinweisen soll.

An dieser Stelle sei auch Herrn Prof.G.Lochs,Innsbruck, für das fördernde Interesse an dieser Arbeit, Herrn Dr.H.Harmuth, Karlsruhe und Milwaukee, für die Anregung und die gewährte finanzielle Unterstützung, sowie den Herren Dr.R.Liedl und Dr.P.Weiß, beide Innsbruck, für gute Ratschläge und zahlreiche Diskussionen herzlichst gedankt.

Innsbruck, im Mai 1967 Franz Pichler

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INHALTSVERZEICHNIS

How I came to Maryland is interesting, because Harmuth was not an easy man, and he is until today not easy, but he is a devoted scientist and so on, so I had some struggle with him in Innsbruck already. But he was so concentrated to push forward Walsh functions and research, and then, I think we split. We had no much contacts, but I continued to make my work, especially the PhD thesis, and also papers. Yes, and in 1968 I think I published my first paper in the AEU (*Archiv* der *Elektrischen Übertragung* abbreviated as *Archiv eiektr. Übertragung),* this is the journal where Hansi Piesch was an editor, or co-editor. I think I mentioned this already once to you. You see, it was not so easy at that time to publish about Walsh functions. AEU had already published the papers of Harmuth, and I was still a student, and not experienced. So I had my doubts if they would accept it, but I had a promoter in the East Germany, in the DDR. This was Franz Heinrich Lange, a Professor of communication engineering in Rostock. He was very well known, and he was fan of Harmuth and of Walsh functions, and so on, and he knew about my work, and when I wrote and sent my paper to the AEU, I do not remember the main editor there, the secretary so to say, I mentioned that if they would not publish it, I could publish it in the DDR, because Lange would have liked to take it. I was already clever, I think, to mention this and they finally reviewed the paper and one of the reviewers was Hansi Piesch. And so I really brought the final version of the paper to Vienna. I drove with my Fiat 600 from Innsbruck, with a friend of mine, we drove to Vienna, and I went to her apartment, yes, at the Gürtel near Sud Bahnhof, and was friendly welcomed and I gave the paper for publication.

We had a chat, and I did not know at that time about her research, and I was also not familiar with the work of Ms. Piesch, for example, in switching functions and so on, at that time I was still a mathematician, with some background in communication engineering. But at that time, I did not study automata theory.

I was very disappointed once, when at the mathematics department there were lectures on automata theory and I did not see the links with what I had experience in my practical work, because were relay switching circuits and this are asynchronous automata. And a lot of delay phenomena are used in switching and so on, and I did not see this in the mathematical framework of automata theory of the course.

I came back and finally I went to Linz in 1968, and Harmuth again contacted me, and needed me for this first conference as a mathematician, because he was always criticized that he could not define exactly the Walsh function in the continuous case of sequency as he called it. Yes, so he needed me, Harmuth was able to define Walsh functions just as a limes, yes, if *n* goes to infinity then this is the function, yes, so he was picked by some people when they said *Tell me, how does the Walsh function with sequency Pi, yes, 3.14 and so on, does look like*?, and he could not answer. He could not really answer, he was, and really is a gifted intuitive working scientist. A kind of engineer with mathematic intellect, Harmuth, there is no doubt yes. Like also Gibbs [1], they would make formulas without knowing how they can derive these formulas. I was just the opposite. I was, say, educated as a step-wise, going further, operating mathematician, so he needed me in Washington, and I got the invitation as a visiting research assistant professor.

1 **J. Edmund Gibbs**, a scholar and mathematician who introduced the notion of a family of differential operators known as the *Gibbs derivatives*.

For the biography of J.E. Gibbs, see F. Pichler, "Remembering J. Edmund Gibbs", in *Walsh and Dyadic Analysis*, R.S. Stanković (ed.), *Proc. Workshop on Walsh and Dyadic Analysis*, October 2007, Faculty of Electronic Engineering, Niš, Serbia, XXI-XXVI, 2008.

F. Pichler (left) and J. Edmund Gibbs (right) with associates in the National Physical Laboratory, Teddington, Middlesex, United Kingdom in 1974.

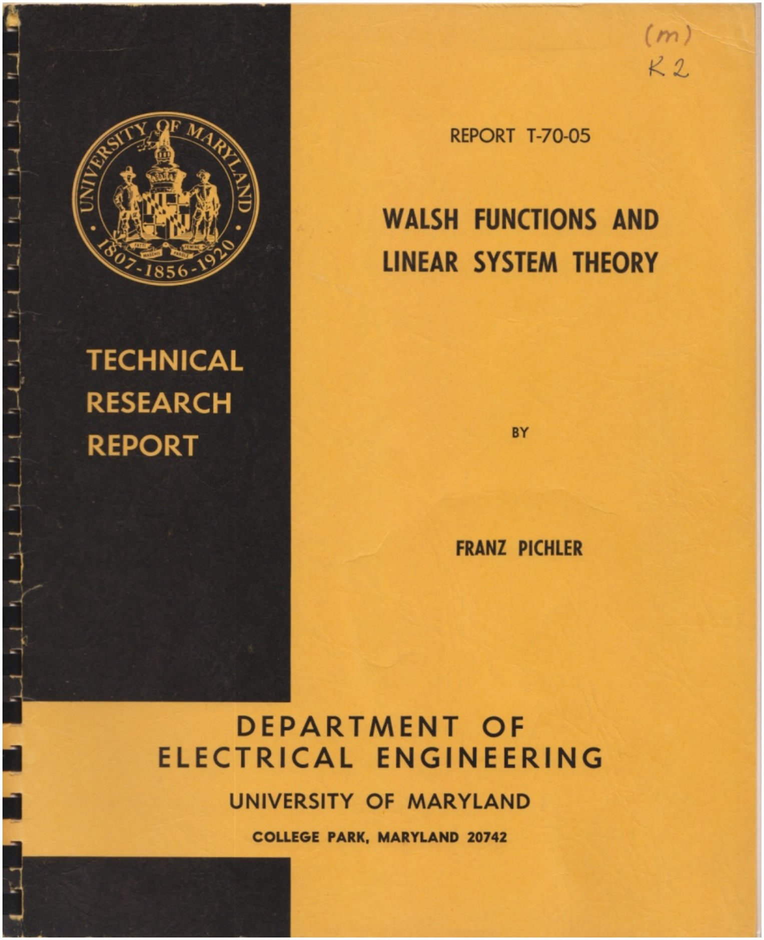
I had a family here in Linz and I was assistant professor at the *Chair of Applied Mathematics*. But my boss, Professor Hans Knapp, he was very supportive and he said *Well, if you have such an invitation, you should go*.

But, in that time I did not take that risk to take the family with me. So the family stayed in Linz, I had to send money every month, and I had even to pay a teaching assistant, because my Professor needed help. So, I had to pay from my salary also a teaching assistant. This was very difficult for me being in Maryland. I went in February of 1970 to Maryland and the semester had already started. And Harmuth as a man, he can make everything, yes, he can teach Electrical Engineering, he can teach laboratory, everything. So when I came to Maryland, only resident of Maryland Electrical Engineering Department, the semester was already running. I was employed for undergraduate teaching but, finally, it turned out that I had to take the courses which were left. So I got a course on Electric and Power Laboratory and I had to run it. This was not my field, I was a mathematician. But, finally, I should make it short, and it was not so difficult as I thought it is and as I was used. I studied Physics in Innsbruck also. So I knew what laboratory is, how to run a laboratory, though they had nothing prepared. They said *We make this every semester, individually again*, and they showed me a heap of junk hardware. The Dean of the Department of Electrical Engineering, Professor DeClaris said, *You can use everything which is here*. So I was confused, you see, frankly, I did not find in the library, say, an electrical engineering book in English there, which could help me. Yes, I needed it, so I was really close to return, to give up, to say this is not my job. So I felt lost, but finally I started, my English was very bad, and, I got seven teaching assistants for help. This was a course for seniors. This was the last in the curriculum, and, it turned out that it was the last Electrical Power Engineering course they made. They gave up then this line, none was interested. But I really have to praise the colleagues, first of all the assistants, seven assistants, there were some which were experienced, they were already involved in former laboratories. I especially like to mention this if you write it, and we should mention him, I remember a Chinese, Mr. Chang, he was a PhD student. He was very good. And he helped me a lot. And also another from Thailand, they were, all seven were from different countries. From Thailand, Mr. Lekiananda, I really remember, and also another from Greece, only one American yes, from India, and he was lazy, he did not help me. But, I should praise the American colleagues because they helped me. So week for week, we had to set up an experiment, and I got advice from the colleagues in the Laboratory from Professor Simon and others. And so, finally I made it. My English improved, and the students, this is important, were happy, But it was a very hard, very hard work, beside I had another laboratory on Signal Processing. This was easier for me, experiments with convolution of ratio and such things, so this was easier, this went quite smooth, but the Power Engineering, this was very difficult. But since I had experience as a telecommunication technician, and have some knowledge in electrical engineering I could do it, otherwise, a normal mathematician cannot do it. You know, even what is called the Prony zaum (Prony brake), this is to measure the power of an electric motor, it is a big machine they had for it, was applied.

And I come back now to Harmuth.

Harmuth, of course, wanted to make Walsh function research, and so I did. This was the main reason that I am there, he had a grant for that, I was paid from this grant. But besides, I had this teaching duties, he did not much teach, finally I have to say, he might not agree, but at the end of the semester, I got more agreement by the colleagues, and acceptance as Harmuth, because he was always individual.
And I was a team worker, so they knew me, and they liked me, and finally this was a very good work and stay. And then came this conference, and of course this was the

main reason, I should say. I wrote some little articles, like on linear systems, and DeClaris said, *Well, make a report*. They wanted to put out some report, so this was immediately taken. It is I think essential the same as that was published also in the conference. Then I wrote this article on this subject. This thick report was on dyadic correlation, and a chapter including the proof of some theorem of Edmund Gibbs. This was, I have to say, to get an idea of Harmuth. Well, the conference went smooth, this was the first conference on Walsh functions, so this was interesting. Some of the younger colleagues who came over for the first time from Europe, like Felix Bagdasarianz from Switzerland, he worked also at ETH Zurich on Walsh functions, and also a PhD student from Aachen, Dieter Roth, he was a good friend of mine, when they found me there, they wanted to stay with me, in the same house. It was easier, they hate hotels, so we had some fun together, these Europeans which came over and for the first time. But Bagdasarianz and I were the only ones who were not from NATO countries. So finally, I think Bagdasarianz found out an interesting detail, he was later in Switzerland a very important man at some company, he is still around in Switzerland. He has an Iranian background from his father's side I think, but he is a Swiss. He found out that they watched us all the time, even when we went to the men's room, toilet, some would come with us, and Butzer, I remember, Butzer wanted to go to the library of Naval Research Lab, and this was impossible. I remember that, he wanted to look at the library, and so they said no, no. We were on a really on a secret place, but otherwise it was really nice and was a good conference. I started to mention, that really in America there were many good scientists already experimental and theoretical which worked on this, like Lechner. I was impressed, I was not such a good mathematician, I was not informed about group theory, and Lechner was. You immediately could see that he was a very good mathematician. Yes, I was more kind of applied mathematician, with not such a background. I was already more engineer in Maryland, and also the others. And also experimental work was done for example by Ohnsorg, he had a German background. And an occasion for discussing this conference was. I knew Joseph Walsh already earlier, because after Harvard he came to Washington D.C. as Emeritus Professor and he was with the Mathematics Department of the University of Maryland there. So he was already there. This was I think just a lucky coincidence, this had nothing to do with Harmuth, but Harmuth was there, Walsh was also there. Walsh was a very nice man, yes, already well in my age now, but he looked much older I think. Joseph Walsh was, I think I mentioned this already before and in my paper also in Niš, he was very happy that now his functions were discovered again outside of mathematics. In mathematics they had a tradition already. But now outside applied mathematics, and so he was very, very pleased, and he invited especially the Europeans which came, there was Gibbs, Butzer, and I think also my friend Bagdasarianz, and the one from Aachen and me at his house. I really could not remember so exactly because we made always already in Innsbruck some fun with the Walsh functions. For example Roman Liedl made a relief with the Walsh functions in color in the Mathematics Institute. Walsh had, I think it was not his tie, but his socks, I think he had socks which were decorated with the Walsh functions and his wife made this for him. So this was, you know that mathematicians always like to be childish, in a positive sense. I think is a kind of a quality mathematicians have. This is what you can find.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REPORT T-70-05

WALSH FUNCTIONS AND LINEAR SYSTEM THEORY

BY

FRANZ PICHLER

TECHNICAL RESEARCH REPORT

DEPARTMENT OF ELECTRICAL ENGINEERING

UNIVERSITY OF MARYLAND

COLLEGE PARK, MARYLAND 20742

Walsh Functions and Linear System Theory*

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* Lecture to be presented at the workshop on "Applications of Walsh Functions", Naval Research Laboratory and University of Maryland April 2, 1970.

CONTENTS

APPENDIX B

Sampling Theorem with Respect to Walsh-Fourier Analysis

1. Introduction

In the following we are concerned with a generalization of the sampling theorem for band-limited signals. The origin of this classical theorem of communications can hardly be traced. In the mathematical literature it, or some of its analogues, is connected to several authors (e.g. Cauchy [1], Whittaker [2]). The same situation seems to be in the field of communications (e.g. Kotelnikov [3], Shannon [4], Raabe [5]).

Here we shall deal with a formulation of a sampling theorem for the case of representing signals as superpositions of Walsh functions. We shall use results of a paper by Kluvànec [6], in which the sampling theorem has been formulated in terms of the theory of abstract harmonic analysis. We specialize the case of an arbitrary locally compact abelian group to the case of the dyadic group of Fine [7] and get the sampling theorem in terms of Walsh-Fourier analysis. It turns out, that this theorem is a very trivial one if one pays attention to the fact that a function, band limited in the sense of Walsh-harmonic analysis, is equal almost everywhere to a stepfunction. A slight modification of this theorem, depending on the use of the Walsh functions sal(s, t) and cal(s, t), has been presented in a former paper of this author [8].

2. Sampling Theorem in Abstract Harmonic Analysis

We follow Kluvànek [6]. Let G be a locally compact abelian group (written additively) and $\hat{G}$ the dual group. The value of a character $y \in \hat{G}$ at a point $x \in G$ will be written as usual as (x, y).

Suppose H be a discrete subgroup of G with discrete annihilator Λ given by $\Lambda = \{ y \in \hat{G}: (x, y) = 1 \text{ for all } x \in H \}$. Let $[y]$ denote the coset of Λ which contains the point $y \in \hat{G}$; i.e. $[y] = y + \Lambda$. Let further the set Ω be defined as a measurable subset of $\hat{G}$, $\Omega \subset \hat{G}$, which contains exactly one point from every coset $[y]$ of Λ, i.e. $\Omega \cap [y]$ is a singleton for all $y \in \hat{G}$.

The Haar measure on G is denoted by m, that of $\hat{G}$ by $\hat{m}$. The Haar measure $[\hat{m}]$ of the factorgroup $\hat{G}/\Lambda$ should be normalized, so that $[\hat{m}](\hat{G}/\Lambda) = 1$. The Haar measure on H and Λ respectively have at each point $x \in H$ and $y \in \Lambda$ respectively the value 1. Then $\hat{m}$ can be normalized, so that

$$\int_{\hat{G}} \hat{f}(y)\, d\hat{m}(y) = \int_{\hat{G}/\Lambda} \sum_{z \in \Lambda} \hat{f}(y+z)\, d[\hat{m}]([y]) \qquad (1)$$

holds for every integrable function $\hat{f}$ on $\hat{G}$. Finally let the Haar measure m on G be adjusted so that the inversion formulas for Fourier transforms holds, i.e. by the relations

$$\hat{f}(y) = \int_G (-x, y) f(x)\, dm(x) \qquad (2)$$

and

$$f(x) = \int_{\hat{G}} (x, y) \hat{f}(y)\, d\hat{m}(y) \qquad (3)$$

Let further the function φ be defined by

$$\varphi(x) = \int_{\Omega} (x, y) d\hat{m}(y) \qquad (4)$$

Due to Kluvanek [6] we have the following lemma and theorem:

Lemma: The function φ is defined for all $x \epsilon G$. It is continuous, positive-definite and belongs to $L^2(G)$. Its norm $||\varphi||$ in $L^2(G)$ is equal 1 and $\varphi(0) = 1$. For all $z \epsilon H$ with $z \neq 0$ we have $\varphi(z) = 0$ and

$$\int_G \varphi(x)\overline{\varphi(x-z)}\, dm(x) = 0 \qquad (5)$$

Sampling theorem: Suppose $f \epsilon L^2(G)$ and $\hat{f}(y) = 0$ for almost all $y \notin \Omega$. Then f is equal almost everywhere to a continuous function. If f itself is continuous then

$$f(x) = \sum_{z \epsilon H} f(z)\varphi(x-z) \qquad (6)$$

uniformly on G and in the sense of the convergence in $L^2(G)$. Furthermore

$$||f||^2 = \sum_{z \epsilon H} |f(z)|^2 \qquad (7)$$

If $G = T = (-\infty, \infty)$, the set of real numbers, and H is given by $H = \{mT: m = 0, \pm 1, + 2, \dots\}$ where $T = 1/2f_o$, then we have $\Omega = (-2\pi f_o, 2\pi f_o)$ and φ is given by

$$\varphi(x) = \frac{\sin 2\pi f_o(x - mT)}{2\pi f_o(x - mT)} \qquad (8)$$

and formula (6) becomes the classical form

$$f(x) = \sum_{m=-\infty}^{\infty} f(mT) \frac{\sin 2\pi f_o(x - mT)}{2\pi f_o(x - mT)} \qquad (9)$$

3. Sampling Theorem in Dyadic Harmonic Analysis

Our intention is now, to formulate the sampling theorem of above with respect to the case that $G = \mathfrak{F}$, the dyadic group of Fine [7]. According to Fine the dyadic group $\mathfrak{F}$ is given by the set of infinite sequences $\bar{x}$ of the form

$$\bar{x} = (x_i) = (\ldots 00x_{-M}x_{-M+1}\ldots x_0x_1x_2\ldots)$$

the components x_i being 0 or 1 and each $\bar{x}\epsilon\mathfrak{F}$ being 0-periodic to the left side. It is helpful to identify the set $\mathfrak{F}$ with the set of binary representations of nonnegative real numbers. The composition $\oplus$ in $\mathfrak{F}$ is defined via addition modulo 2 of the components. $(\mathfrak{F},\oplus)$ is then obviously an abelian group, each $x\epsilon\mathfrak{F}$ having order 2; $\bar{x}\oplus\bar{x} = 0$. A topology can be found, such that $(\mathfrak{F},\oplus)$ becomes a locally compact topological group [7], [9]. The character group $\hat{\mathfrak{F}}$ of $\mathfrak{F}$ is algebraically isomorph to $\mathfrak{F}$. So each character $\bar{y}\epsilon\hat{\mathfrak{F}}$ can be represented by a sequence of the form

$$\bar{y} = (y_i) = (\ldots 00y_{-N}y_{-N+1}\ldots y_0y_1\ldots)$$

The value $(\bar{x},\bar{y})$ of a character $\bar{y}\epsilon\hat{\mathfrak{F}}$ at a point $\bar{x}$ is given by

$$(\bar{x},\bar{y}) = \exp \pi i \sum_{i+k=1} y_i x_k \qquad (10)$$

From (10) we can see that the characters $\bar{y}\epsilon\hat{\mathfrak{F}}$ are real valued functions which maps $\mathfrak{F}$ onto the set $\{+1,-1\}$. The Haar measure on $\mathfrak{F}$ and $\hat{\mathfrak{F}}$ respectively should be normalized such that the subgroup $\mathfrak{D}$ and $\hat{\mathfrak{D}}$

consisting of all sequences $\overline{x} = (x_i)$ and $\overline{y} = (y_i)$ respectively, where $x_i = y_i = 0$ for all $i \le 0$ have the measure 1. We can now establish the sampling theorem for the dyadic group $\mathfrak{F}$. The subgroup H should be given by $H = \{\overline{x} \in G: x_i = 0 \text{ for all } i > k\}$, where k is an integer. Then Λ is given by the subgroup $\Lambda = \{\overline{y} \in \hat{G}: y_i = 0 \text{ for all } i > -k\}$. The set Ω consists of all sequences $\overline{y} \in \hat{\mathfrak{F}}$ with $y_i = 0$ for all $i \le -k$. The functions φ we define (4) by

$$\varphi(\overline{x}) = 2^{-k} \int_{\Omega} (\overline{x}, \overline{y})\, dm(\overline{y}) \qquad (11)$$

The difference of (11) to (4) comes from the fact, that we normalized the Haar measure m so that $m(\Omega) = 2^k$ rather than 1. Integration gives us

$$\varphi(\overline{x}) = \chi_{1/\Omega}(\overline{x}) \qquad (12)$$

where $\chi_{1/\Omega}$ denotes the characteristic function of the set $1/\Omega$ given by $1/\Omega = \{\overline{x} \in \ \ | x_i = 0 \text{ for all } i \le k\}$. Therefore formula (6) of the sampling theorem gets the form

$$f(\overline{x}) = \sum_{\overline{z} \in H} f(\overline{z})\, \chi_{1/\Omega}(\overline{x} \oplus \overline{z}) \qquad (13)$$

4. Sampling Theorem in Walsh-Fourier Analysis

Walsh-Fourier analysis is a Fourier analysis for functions defined on the nonnegative real line $[0, \infty)$. It can be derived from dyadic Fourier analysis in the following way: The map $\lambda: \mathfrak{F} \to [0, \infty)$ is according to Fine [7] defined as the map, which takes a point $\overline{x} = (x_i) \in \mathfrak{F}$

into a point $x \in [0, \infty)$ given by

$$x = \sum_i x_i 2^{-i} \qquad (14)$$

For $x \in [0, \infty)$ the inverse mapping μ is defined by (11) choosing the finite dyadic representation if x is a dyadic rational. The map μ neglects only a set $\mathcal{E}$, consisting of sequences which are 1-periodic to the right side; μ: $[0, \infty) \to \mathcal{F} \setminus \mathcal{E}$. The set $\mathcal{E}$ has Haar measure zero.

So we have

$$\lambda(\mu(x)) = x \qquad \text{for all } x \in [0, \infty) \qquad (15)$$

and

$$\mu(\lambda(\overline{x})) = \overline{x} \qquad \text{for all } \overline{x} \in \mathcal{F} \setminus \mathcal{E} \qquad (16)$$

he mappings $\hat{\lambda}$: $\hat{\mathcal{F}} \to [0, \infty)$ and $\hat{\mu}$: $[0, \infty) \to \hat{\mathcal{F}} \setminus \hat{\mathcal{E}}$ where $\hat{\mathcal{E}} \simeq \mathcal{E}$, are defined in a analogous way as the maps λ and μ.

The Walsh functions $\psi(y, \cdot)$: $[0, \infty) \to \{+1, -1\}$ are defined as the functions given for all $x, y \in [0, \infty)$ by

$$\psi(y, x) = (\mu(x), \hat{\mu}(y)) \qquad (17)$$

and the transforms formula given by (2) and (3) becomes the form

$$\hat{f}(y) = \int_0^\infty \psi(y, x) f(x)\, dx \qquad (18)$$

and

$$f(x) = \int_0^\infty \psi(y, x) \hat{f}(y)\, dy \qquad (19)$$

where $f \in L^2[0, \infty)$. This comes from the fact that the map λ^*: $P(\mathcal{F}) \to P([0, \infty))$, which maps the power set $P(\mathfrak{F})$ of $\mathfrak{F}$ into the power set $P([0, \infty))$ of $[0, \infty)$ and which is induced by λ, is a Haar-Lebesgue measure preserving map. The same is true for $\hat{\lambda}*$. From that it turns out, that the algebraic and measure theoretic results of the sampling theorem formulated above for $G = \mathfrak{F}$ are invariant against the map $\lambda*$ and $\hat{\lambda}*$. However, the topological results concerning continuity etc. are altered. We have so

$$\lambda*(\mathfrak{F}) = [0, \infty), \quad \hat{\lambda}*(\hat{\mathfrak{F}}) = [0, \infty),$$

$$\lambda*(H) = \left\{ m2^{-k}: \ m = 0, 1, 2, \ldots \right\}$$

$$\hat{\lambda}*(\Lambda) = \left\{ n2^{k} : n = 0, 1, 2, \ldots \right\}$$

$$\hat{\lambda}*(\Omega) = [0, 2^{k}), \quad \hat{\lambda}*(1/\Omega) = [0, 2^{-k})$$

$$dm(\overline{x}) = dx$$

$$dm(\overline{y}) = dy$$

The function $\lambda* \varphi$: $[0, \infty) \to R$ given for all $x \epsilon [0, \infty)$ by

$$\lambda*\varphi(x) = 2^{-k} \int_{0}^{2^k} \psi(y, x)dy \tag{20}$$

is the characteristic function $\chi_{[0, 2^{-k})}$ of the interval $\lambda*(1/\Omega) = [0, 2^{-k})$. From above the sampling theorem becomes for Walsh-Fourier analysis the form:

<u>Theorem</u>: Suppose f is a real valued function of the space $L^2[0, \infty)$ and $\hat{f}(y) = 0$ for almost all $y \notin [0, 2^k)$. Then f is equal almost everywhere to a stepfunction, continuous from the left which jumps only at points x of the form $x = m2^{-k}$, $m = 0, 1, 2, \ldots$. If f itself is of that kind then

$$f(x) = \sum_{m=0}^{\infty} f(m2^{-k}) \chi_{[0, 2^{-k})} (x - m2^{-k}) \qquad (21)$$

uniformly on $[0, \infty)$ and in the sense of the convergence in $L^2[0, \infty)$.

Furthermore

$$2^k ||f||^2 = \sum_{m=0}^{\infty} f^2(m2^{-k}) \qquad (22)$$

Knowing that f is equal almost everywhere to a stepfunction of that kind described above, this theorem is a trivial one. It is obvious that such a stepfunction can be generated by characteristic functions as shown in equation (21).

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I tell you this just as a folklore, it is nothing to read, but Harmuth of course was an adventurer. And his adventure was to go with a kayak, with a boat, on wild rivers. And he did this also in Innsbruck, coming down the river Inn, I went with him, I said 'hello' when he arrived, and so he also took me. Also, just in the beginning of my stay in Maryland, we went to the Potomac River in Washington, and he put me in a kayak. I had no experience to go out and, of course, I had problems. Then, the kayak turned over and I swam in the Potomac River, and also the kayak. And he was, I think, he was worried about the kayak, not what had happened to me. I was close to have real troubles. It was quite unpleasant, because I was so nervous, and I really had problems, so I finally got the kayak, and I saved me and the kayak on a little island, so we were safe. And that way, he got angry, so Harmuth is not an easy man, he is really a strange man, in that situation, yes, on the other hand he is very, very, nice also, but in some situation he has no patience. Then it was an idea I should join him when he wanted to go to Canada, to cross in the northern part from Winnipeg, going east from one river to another river, with a kayak. And my task was only to go with him to Winnipeg, and drive the car back. This is a long way. Because he did not need the car then, and he was on his own I think, at that time. It was the plan that he goes alone, but maybe he had already a friend to be together with him because alone is not possible, and I was selected, this was my duty. And then, we went to the south of Washington, to Cape Hatteras. This was a nice trip and he made me drive, like to test if I am a good driver. At that time, him and I already made an agreement and it was so that I had to stay longer, also for the summer term, and he would use this for that trip. I could go with him, I was also a kind of adventurer. My wife Ilse was disappointed that I did not return in May to Linz and stayed longer until August. It was then so that, finally, he was so nasty to me, in driving and criticizing and said, no, I cannot use you for this as a driver. You cannot drive my car back, and he cancelled this agreement due to his disappointment with me. But I already had agreed to stay longer, it was already a contract and other related things. Then, he went with a colleague and they had a very hard time, because you have to carry the boat sometimes, up to high sights, and to find another river, and then go down again. So this was unpleasant, I have known this friend of Harmuth, their friendship broke by that, he did not speak anymore to Harmuth later on, because he was on the end of him. But Harmuth was of this type, he was an adventurer. Like Reinhold Messner. He could have climbed also the very Mount Everest, and so he is a strong man, he was really active, and clever.

So I had to stay and then he said *Well, you have to make research and so on*. Yes, they wanted to see then research, something like that, and he gave me a task and this was the correlation paper and also the proof of the theorem of Gibbs. It was so, he cannot estimate how quick a mathematician works, I think I had everything in two weeks. So I had done my duty, no one was around, but I had some good friends at the University so it was finally also a good time. I went to Chesapeake Bay, this is an area east of Washington, and there was also a colleague of mine there, an Indian, graph theorist, this is Ramanujacharyulu was his name, and they called him only Chary. He was formerly with Bell Laboratories and he was also an Assistant Professor, Dr. Chary, we got a friend and also others. So this was finally a good time and Harmuth was happy when he came back, and had this thick report and this was enough. He was then not so critical, and we had nice periods in meeting him, but there were also some problems. His wife was very nice, yes, he had a very, very nice wife which he was divorced later, and so on. It is so that later we were in good, good contacts and at later meetings, it was not so hard. Somehow he got older and a little bit more human. So I finally had no problem with him and I made also some lectures about his work. You know I have given some lectures in honor of Harmuth, and also others like Steinmetz, Küpfmüller and Cauer. I lectured once at a symposium, this was published, and we were in good standing.

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Karl Küpfmüller - a German electrical engineer, known by his work in communications technology, measurement and control engineering, acoustics, communication theory and theoretical electro-technology.

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Paul Leo Butzer, a well renowned mathematician from Rheinische-Westfaelische Technische Hochschule (RWTH) Aahen, Germany, also widely recognized by his work in Walsh and dyadic analysis. The Butzer-Wagner derivative is a very important notion in this area.

ON STATE-SPACE DESCRIPTION OF LINEAR DYADIC-INVARIANT SYSTEMS

by

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1. Introduction

In this paper, we try to describe linear dyadic-invariant systems with the aid of the concept of a state space. Up to now, little or no attention has been paid to this question. Consideration has usually been restricted to systems of convolution type, and, in particular, to generalized frequency descriptions of such systems.

Our concept of a linear (finite, discrete) dyadic-invariant time-system is nothing other than that of a special discrete linear constant system $[\underline{A}, \underline{B}, \underline{C}, \underline{D}]$, which is described in the following well-known form:

$$\underline{x}(t+1) = \underline{A}\,\underline{x}(t) + \underline{B}\,\underline{u}(t), \qquad \underline{y}(t+1) = \underline{C}\,\underline{x}(t) + \underline{D}\,\underline{u}(t). \tag{1}$$

We get the special structure of such a system that we want by the assumption that the codomain U of the input time-functions, the codomain Y of the output time-functions, and the state space X are all normed commutative algebras.

In Section 2, we give a brief introduction to the mathematics involved, with emphasis on matrix notation. Section 3 shows that any dyadic convolution operator is brought into diagonal form by Walsh-Fourier transformation. Sections 4-6 are concerned with dyadic-invariant systems, and state-space descriptions of such systems.

A major purpose of this paper is to show that dyadic-invariant systems can be embedded into the familiar concept of linear discrete constant systems. Further, the description of dyadic-invariant systems requires only elementary tools of linear algebra.

2. Dyadic algebras

Let B(n) denote the finite discrete dyadic group consisting of the set of all binary numbers x of the form $x = x_{n-1} \dots x_o$, together with the group operation $\oplus$ given by component-wise addition modulo 2. Let N(n) denote the set of all non-negative integers less than 2^n. Let λ be the bijection λ: B(n) → N(n) which is given by

$$\lambda(x_{n-1} \dots x_o) = x_o 2^o + \dots + x_{n-1} 2^{n-1}$$

for all $x_{n-1} \dots x_o \in B(n)$. We shall regard N(n) as isomorphic to the dyadic group B(n), the isomorphism being represented by λ. The addition in N(n) thus defined we denote also by $\oplus$. We denote by L the linear space of all real-valued functions defined on N(n). The set $E = \{e_i : i \in N(n)\} \subset L$, where the functions $e_i \in L$ are given by $e_i(j) = \delta_{ij}$ for all $i, j \in N(n)$, (δ_{ij} is the Kronecker symbol), is a basis for L. This basis E will be referred to as the standard basis of L. Any function $f \in L$ can be represented with respect to the basis E as a linear combination of the form

$$f = \sum_{i \in N(n)} f_i e_i ,$$

or as a column vector $\underline{f}$ of the form

$$\underline{f} = \begin{bmatrix} f_0 \\ \cdot \\ \cdot \\ \cdot \\ f_{2^n-1} \end{bmatrix} . \tag{2}$$

We now define in L some operations that are useful. First, we make L into a Hilbert space (L, <,>) by defining an inner product <,>: L×L → $\mathbb{R}$ in the usual way, using the basis E:

$$(f, g) \mapsto \langle f, g\rangle = \sum_{i \in N(n)} f_i g_i \quad (f, g \in L). \tag{3}$$

The norm $\|\cdot\|$ of L is given by $\|f\| = \langle f, f\rangle^{1/2}$. In (L, <,>), the standard basis E represents a complete orthogonal and normal set of functions.

Next, we define in L a generalized translation: for arbitrary $i \in N(n)$, let $\sigma(e_i)$ denote the linear operator $\sigma(e_i)$: L → L that is given by

$$f \mapsto \sigma(e_i)f = \begin{bmatrix} f_{0 \oplus i} \\ \cdot \\ \cdot \\ \cdot \\ f_{2^n-1 \oplus i} \end{bmatrix} . \tag{4}$$

We call $\sigma(e_i)$ the i-th dyadic translation operator. The set $\sigma E = \{\sigma(e_i): e_i \in E\}$ of all dyadic translation operators forms with respect to composition an abelian group, which we call the dyadic translation group.

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Franz Pichler, "Walsh-Functions: Early ideas on their application in Signal Processing and Communication Engineering", *Proc. 2004 International TICSP Workshop on Spectral Methods and Multirate Signal Processing*, SMMMSP 2004, Vienna, Austria, September 11-12, 2004, Edited by Jaakko Astola, Karen Egiazarian, Tapio Saramaki, ISBN 9S2-15-1229-6, pp.263-269, 2004.

I found out that the American system of mobile telephony, as developed by the Qualcomm company uses the Walsh functions heavily. More than the GSM and then I asked how does it come. They use it really as we decided. But we were in the analogue world, and then in the digital it went different, but it was the same idea. And then it came out that one of the founders of Qualcomm is Professor Viterbi 1 and he had also, say, experience with Walsh functions. He knows Walsh functions from the Jet Propulsion Laboratory where he worked. Golomb 2 is another one, then Viterbi, they were familiar and also Dertouzos 3, but also Chow 4. I met Chow, and when I said to him Well, you are the Chow with the Chow parameters, or something like that, he was very pleased.

1 **Andrew James Viterbi**, Ph.D., is an Italian-American Electrical engineer and businessman.

2 **Solomon Wolf Golomb** is a mathematician and engineer and a professor of electrical engineering at the University of Southern California

3 **Michael Leonidas Dertouzos** was a Professor at the Massachusetts Institute of Technology and Director of the M.l.T, Laboratory for Computer Science (LCS) from 1974 to 2001.

4 **C.K, Chow**, "On the characterization of threshold functions", in *Proceedings of the Symposium on Switching Circuit Theory and Logical Design* (*FOCS*), 1961, 34-38.

Professor Pichler spoke about Professor Heinz Zemanek

Heinz Zemanek first was a docent at the Technical University of Vienna, and then an Associate Professor at the same University. Then he built the Mailufterl computer in the fifties, and by the publications, and by this fact, IBM took notice of this group in Vienna, the Zemanek group, and, they hired the group, the whole group, and a laboratory was founded. Sometimes I think IBM just wanted to control the development of computers in Vienna at that time. I do not know how exactly, but the group then completely changed the field. Before it was hardware oriented, now they became software oriented and famous people worked with Zemanek, like P. Lukas, V. Kudielka, and others. I think one of the great things they made is the formal definition of PL1. You see, this was once an important language, IBM promoted it, we had it also as the standard language in lecturing. And, as usually these language designers did not design it top-down. The PL1 language is a result of practical evolution, and then to find a formal definition, and especially also the semantic of the language, this was a hard work. And by then in Vienna, the Vienna Definition Language was developed, this is still in Computer Science known, because it was a new method of defining the semantic and of a language, and I think they were very successful. Finally, the laboratory had to close, as usual, after some years. Zemanek, got a special shop and became a, say, special grade at IBM (IBM fellow). You can read about that. Then, he concentrated to research on computer architecture, but from a very systems engineering point of view. So architecture means a lot to Zemanek. There are many books, also articles written by him. So, this is in short the history of the Zemanek work at that time. But he continued of course to write and give lectures, and was active.


2
Taschenbuch
der Informatik


HEINZ ZEMANEK
SAAL
OESTERREICHISCHE
COMPUTER GESELLSCHAFT
AUSTRIAN
COMPUTER SOCIETY

I think it is very nice that we still can get this interview from him. He says he will not give any interviews any more, because this is too time consuming, it brings him back from making the second edition of his calendar book. Zemanek is right now and already for a long time interested in the calendar science and he has written a book on that. I have to confess that I did not look at it, but this is what keeps him, say, active. I had many meetings with him, he is a honorary doctor of the University of Linz, is a honorary doctor with a degree of our university, and this is not by me, there are many colleagues who know him much better than I, but he is very critical, and I can be very glad that he has a good meaning of me and of my work. So, he thinks that everything outside of Vienna is easier.

Zemanek says you are doing good things in Linz and at the Tesla museum, but he does not know that I am just so isolated as he is in Vienna. You just have to do everything on your own, so this is the situation in reality.

TESLA MUSEUM GALLSPACH

Erlebnis „Elektrizität"

Franz Pichler

Einführung

Im Dezember 2006 wurde aus Anlass des 150sten Geburtstages von Nikola Tesla, der wohl zu den wichtigsten Erfindern in der Elektrotechnik zu rechnen ist, in Gallspach, OÖ, der Verein „Österreichische Tesla Gesellschaft" gegründet. In diesem Zusammenhang entstand auch der Plan zur Einrichtung eines kleinen speziellen Museums, in dem allgemein das Gebiet der Elektrizität, als Teilgebiet der Physik, und deren Anwendungen im Gebiet der Technik und der Medizin dargestellt werden sollten. Dabei sollten die Erfindungen von Nikola Tesla besondere Beachtung finden.

Die Gründung der Österreichischen Tesla Gesellschaft erfolgte auf Vorschlag von Herrn Medizinalrat Dr. Valentin Zeileis, dem Leiter des Instituts Zeileis in Gallspach. Dieses Institut führt in seiner ärztlichen Praxis neben anderen modernen medizinischen Behandlungen seit mehr als 70 Jahren erfolgreich die Behandlung von Patienten mit „Tesla Strömen" durch, so dass Gallspach in Bezug auf Tesla eine besondere Stellung in Österreich und in ganz Europa hat. Die Österreichische Tesla Gesellschaft und auch das Tesla Museum Gallspach sind jedoch davon unabhängige eigenständige Gründungen. Das Institut Zeileis zusammen mit der Österreichischen Gesellschaft für Informatikgeschichte (ÖGIG) in Wien und die Johannes Kepler Universität (JKU) in Linz sind wichtige Kooperationspartner.

Nikola Tesla : Stationen seines Lebens

Mit Nikola Tesla können vier Nationen bzw. Länder genannt werden. Einmal ist es die serbische Nation und das damalige Königreich Serbien. Nikola Tesla gehörte von seinen Eltern her der serbischen Nation an. Im Alter bekam Nikola Tesla vom Land Jugoslawien eine finanzielle Unterstützung in Form einer Rente. Des weiteren ist Kroatien zu nennen, damals zum Kaisertum Österreich, später zur k.u.k. Monarchie Österreich-Ungarn gehörig, wo Nikola Tesla im kleinen Ort Smilian geboren wurde. Die k.u.k. Militärbehörde in Agram, dem heutigen Zagreb, gewährte ihm für sein Studium in Graz ein Stipendium. Neben Serbien, Kroatien und Österreich sind es die Vereinigten Staaten von Amerika, das Land in dem Nikola Tesla fast sechzig Jahre gelebt hat und in dem er auch zum Erfinder von internationaler Geltung geworden ist.

Das Land Jugoslawien hat nach dem Tod von Nikola Tesla 1943 seinen wertvollen Nachlass erhalten und 1952 in Belgrad das Nikola-Tesla-Museum mit Schauräumen, in dem die Erfindungen von Tesla vorgeführt werden können, zusammen mit einem reichhaltigen Archiv eingerichtet. Damit stellt das Nikola-Tesla-Museum in Belgrad international die für das Andenken an Nikola Tesla und für die historische Forschung zu Nikola Tesla wichtigste Einrichtung dar. Das TESLA MUSEUM GALLSPACH rechtfertigt sich dadurch, dass Nikola Tesla in Österreich sein Studium absolviert hat und dass mit dem Institut Zeileis in Gallspach seit vielen Jahren eine enge Verbindung zum Werk von Nikola Tesla im Gebiet der Elektromedizin vorhanden ist. In den Vereinigten Staaten von Amerika scheint bis heute kein Museum, das Nikola Tesla und seinem Werk gewidmet ist, zu existieren.

Tesla Museum Gallspach: Bereitstellung der Mittel zur Gründung

Das Projekt zur Einrichtung des TESLA MUSEUM GALLSPACH (TMG) stellte eine Initiative der Österreichischen Tesla Gesellschaft dar. Die Bereitstellung der Infrastruktur und die Realisierung des TMG ist nur auf Grund der vorhandenen großzügigen Kooperation mit privaten Personen, Firmen und Institutionen sowie mit befreundeten Vereinen möglich geworden. Von Medizinalrat Dr. Valentin Zeileis wurde aus seinem privaten Besitz die Räumlichkeiten in Form eines Hauses dafür zur Verfügung gestellt. Die ausgestellten Objekte entstammen zum großen Teil aus der Sammlung „RM4IT" (Reales Museum für Informationstechnologie) der Österreichischen Gesellschaft für Informatikgeschichte (ÖGIG). Diese Sammlung wurde ursprünglich am Institut für Systemwissenschaften der Johannes Kepler Universität unter der Leitung von Univ. Prof. Dr. Franz Pichler im Rahmen von Drittmittelprojekten aufgebaut und steht nunmehr vertraglich der ÖGIG als Dauerleihgabe zur Verfügung. Die zur Schaustellung der Objekte erforderlichen Glasvitrinen wurden von der Firma Fabasoft für die Ausstellung „Bausteine der Informationstechnologie" an dem seinerzeitigen Firmensitz im Schloss Puchenau angeschafft und waren später auch bei deren Fortsetzung dieser Ausstellung im

Emer. O. Univ.-Prof. Dr. Franz Pichler lehrte Systemtheorie am Institut für Statistik und Informatik der Universität Linz und gründete 1979 das Institut für Systemwissenschaften an der Universität Linz

IT Center des Softwareparks Hagenberg in Verwendung. Sie stehen nun als Leihgabe dem TMG zur Verfügung.

Abb. 1: Tesla Museum Gallspach

Es ist verständlich, dass das TMG keine originalen Objekte, die von Nikola Tesla stammen, zeigen kann. Keines der seinerzeit vorhandenen Prototypen von Geräten, wie solche für die Einreichung der Patente notwendig waren, hat sich erhalten. Auch sind die von Nikola Tesla fertig gestellten Apparate und Maschinen nicht mehr vorhanden, so dass auch im Tesla Museum in Belgrad nur Nachbauten gezeigt werden können. Für das TMG stehen derzeit noch keine Nachbauten dieser Art zur Verfügung, solche sind jedoch für die Zukunft geplant. Im Gegensatz zum Tesla Museum in Belgrad kann sich das TMG zur Erfüllung der gestellten Bildungsaufgabe nicht auf das Werk von Nikola Tesla allein beschränken. Es muss zusätzlich auch das Gebiet der Elektrizität als Teilgebiet der Physik und auch die Elektrotechnik, hier besonders das Gebiet der Informationstechnologie (IT) mit eingeschlossen werden. Dies ist auch mit dem Untertitel „Erlebnis Elektrizität" im Namen des TMG, ersichtlich. Etwa zwei Drittel der Räumlichkeiten des TMG sind deshalb der Elektrizität und der Elektrotechnik gewidmet.

Führungen

Der Besuch des Museums ist nur im Rahmen einer Führung nach Anmeldung möglich. Es werden drei verschiedene Arten von Führungen angeboten.

1. Führung für Schulen ab 3. Klasse Volksschule für alle Schultypen

2. Führung für Patienten des Institutes Zeileis

3. Führung für ein allgemeines Publikum

Die Führungen werden von dafür speziell geschulten Personen durchgeführt. Zu Beginn stehen dafür auch Mitglieder der „Österreichischen Tesla Gesellschaft" zur Verfügung.

Die Dauer einer Führung beträgt ca. eine Stunde

Organisation des Museums

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Can you tell us please when you started work in Walsh analysis and who motivated you?

OK. So I started to look at Walsh analysis in graduate school because my major advisor [1] had heard that I read Russian. He had proved a theorem about uniqueness of Walsh series and he knew that the Russians (actually, the Armenians-precisely, Arutunyan and Talalyan) had proved a similar result but he didn't know the details. So he wanted me to read it and tell him what was in that paper. He didn't read Russian but I did. So that's the way I got started. I had taken four years of Russian in college so it was almost like a second major (after math).

And, why, how you decided to take the Russian course?

Yes, I was raised in a section of Los Angeles where there was a Russian Orthodox church and a big group of Russians lived there. For example, our landlord was named Semyonov. So I would hear the language as a kid and when I got to college I decided to take Russian as my foreign language. I did this because I had gone to a weak high school and I wanted to be sure that other students who had gone to better schools would not have an advantage over me in my language class. In those days few high schools offered Russian. So I took Russian for two reasons: I had heard it growing up and I wanted to be sure that I wouldn't be at a disadvantage because of my weak high school.

1 Victor Lenard Shapiro

The reason why Shapiro was interested in Walsh series is because he was a student of Zygmund. Zygmund got interested in Walsh series during the forties when he was bouncing around small colleges in the northeast. As you know, he left Poland in 1939 to escape the Nazis and jobs were hard to get then in America. He first taught at Mount Holyoke, a small women's college in Massachusetts for four or five years and then moved on to the University of Pennsylvania, a well respected Ivy League school which didn't have a huge graduate program in mathematics. And so while bouncing around at smaller programs he got interested in Walsh series. In the fifties, after he had moved to the huge program at the University of Chicago, he pursued his second interest, multidimensional trigonometric series, almost exclusively after it took off, but still assigned some graduate students problems in Walsh series, for example, Morgenthaler.

But you know, he still retained his curiosity about Walsh functions. In 1969 or 1970 he told me that he thought that the reason not much had yet been done in Walsh series is because no one had asked the right questions. And of course, shortly after that people began to ask the right questions and the theory of Walsh series began to diverge from trigonometric series. As you know, before 1970 most of the results in Walsh series were analogues of older trigonometric results, but after 1970 we began getting new results which were quite different from the trigonometric case.

One of the things he always maintained is that as the theory of Walsh series matured, new results might be discovered there first which could shed light on the trigonometric case. And, of course, that has been happening over and over again.

There was even an example in the talks today where Fridli and Mandy got a multiplier result first in the Walsh case, because the Walsh case is almost always simpler to look at, and then saw how to obtain a trigonometric analogue.

(The talk given at the Workshop dedicated to the anniversaries of Schipp and Simon)

So, what was the real reason that you determined to devote your work to this area exactly? The beauty of Walsh functions, or simplicity, or what was that?

I liked them, yes. I liked them because they had connections with probability theory which I had studied in graduate school. I also liked them because of their simplicity. There was no need to study deeply into meromorphic functions or differential equations to do something in Walsh series. And finally, I liked them because they did not yet have a long history. I was getting in on the ground floor of a whole new area.

You mentioned this initial paper that your supervisor gave you, what was the name of the supervisor?

Victor Shapiro. A student of Zigmund. He studied at Chicago same time Stein did. And Guido Weiss was a little later in early fifties.

Very nice. And after that, what was the most important publication that influenced you in this work?

Whose publications? Well, my goodness, you are talking about a huge field.

Fine or Mergenthaler or maybe some Russian authors or?

Well, Fine got started, Schipp has done a lot of good things-1 mean there are just so many people who have worked in the field.

How did you begin connections with professor Schipp and the Hungarian group of mathematicians?

OK, so in 1983, I published a survey article about Walsh series. You see I am a taxonomist by nature. You know what that means. I like to collect things and I categorize things. I collect stamps (as does Simon). I collect oriental rugs, old maps, and other things. So I started to collect Walsh papers. In 1971, after I had been scooped on almost everywhere convergence of Walsh-Fourier series of L^p functions (I proved the result but a referee informed me that Hunt and Sjolin had already done it), I decided I needed to know more about what other people had done so I wouldn't prove another known result. I went over to our library and I started looking up every paper on Walsh series. After I caught up to the present, I continued to record and keep a list like that through 2003, but now that I've gotten older I'm just too tired and busy to keep it up. Now not only did I look up every paper on Walsh series in those days and read enough to know what the main results were and what techniques they used, I also wrote summaries of them and actually had copies of almost everyone of them.

So I published this information, as much as I had learned by 1982, in a huge survey article in the International Journal of Mathematics which appeared in 1983. It just so happened to be at the time when Schipp and Simon were talking about writing a book on Walsh series. And so they went, well here is a guy who has a bigger perspective than we do. You know, they had deeper stuff than I did, but they did not have as broad a view as I had, and so they invited me over to Budapest to write a joint proposal with them to ask the NSF, the (American) National Science Foundation, to support our efforts, for example to pay for our travel back and forth and local living expenses, while we worked on this book together. We wrote the grant and got it, and we spent the next six years happily writing a treatise on Walsh series.

This was the first time I met Schipp but of course I knew of him before. He had been doing great work since 1966 and I had some of his papers in my summaries, but I met him in person for the first time in the Fall of 1983.

But your cooperation or your following of the Russian work, of Soviet Union work was much earlier.

That is correct. I had a Fulbright professorship there and it was natural that I should be connected with them first because I was interested in uniqueness of Walsh series and Skvortsov and Talalyan, and several other people in the old Soviet Union, had worked on uniqueness problems. So I applied for a Fulbright and went to Moscow for a Fall Semester, 1977-78. That's where I first met Skvortsov and we wrote a joint paper on the dyadic derivative.

Now when I went there, you know, they were such leaders in this field and I could not very well tell them anything about their own work, so I looked over what had been recently done in the field and realized the one thing they didn't know much about was the dyadic derivative. And so I studied the Butzer-Wagner papers and Schipp's papers on dyadic differentiation and that's what I lectured on when I was there, to give them something new. And then at the end of my stay, Skvortsov and I wrote that paper about dyadic differentiation obtaining very weak conditions, on the dyadic derivative, sufficient to conclude that that a function was constant. So that's how I got over there and how I began to work on the dyadic derivative.

How long did you stay in Moscow?

I was there for five months, one semester, September through January.

Which university was it?

Moscow State.

And later on you wrote a nice review of the book by Golubov, Efimov, Skvortsov, and it is actually a historical review of the development of the dyadic analysis?

Yeah, it is a good book, it is. It is different from ours, ours had a lot of groundbreaking things, theirs was more of a summary, but theirs had more applications, and theirs also considered the more general case of what we call the Vilenkin systems, they do not call them that there, they call them product systems, so I guess they could not very well call them Vilenkin, since Vilenkin was the co-author of that book, right.

And when you mention Vilenkin, in Western literature, they say Chrestenson transform, but actually, Vilenkin was the first one.

Well, you know chronologically, Vilenkin was first. In fact, even before Fine introduced the dyadic group in 1949, Vilenkin had introduced the much more general Vilenkin groups in 1947, But in those days we were very isolated, the west from the east, so we did not really know much about what was going on over there and they did not know much about what was going on here, so that was a problem. I guess that was one of the things that kept me ahead of everyone else after the Russian journals began to appear in our libraries. In those days we had to wait 2-3 years to see translations from Russian to English, but because I was fluent in Russian, I could read them as soon as they came out, and so that helped me a great deal.

That is interesting what you did, you made a bridge between two blocks. Then, ok, you said it was uniqueness and then dyadic derivative?

Yes.

But what else motivated you to start in this harmonic analysis in general, and Walsh functions?

For me? I started out with my dissertation which got uniqueness for almost everywhere convergent Walsh series (previous results supposed that the series converged off a countable set but I discovered that all one needs is that the *lim sup* is finite off a countable set). Then I started looking at uniqueness of summable Walsh series, a natural generalization. Crittenden and Shapiro had a result there, but it wasn't very satisfying. That turned out to be a hard problem, so I started looking at the Haar analogue which I did solve. This brought the problem to Skvortsov's attention and he got the harder uniqueness result for summable Walsh series. After that I published several papers on sets of uniqueness, all the while just investigating related problems as they occurred to me. Then I got interested in the dyadic derivative, as I said before, to lecture in Moscow on something they didn't know, and this interest resulted in several papers too.

Then I got interested in Cesàro summability, not uniqueness this time but convergence, especially the two-dimensional case. The end result of that investigation was the much cited paper joint with Moricz and Schipp on summability of multiple Walsh-Fourier series. They did most of the work, I just posed the problem and started it out by decomposing the multiple Fejér kernel, and then they saw how to continue from there. I originally posed the problem because my advisor had pointed out a 1937 paper by Marcienkievich where he had used the cone condition to prove almost everywhere summability of multiple trigonometric Fourier series. The cone condition: Keep those indices away from the axes and almost everywhere summability works. Anyway, that's what got me to thinking about it. So I did quite a bit on uniqueness, then the dyadic derivative, and then summability, and then finally in the end I came back to uniqueness and got one of the most general results there, the result of a lifetime of thinking about that one issue.

What is your opinion about importance of this concept of dyadic derivative?

Dyadic Derivative. I remember talking to Vilenkin about this when I was living in Moscow. After one of my talks, he said [I do not believe in this derivative speaks in Russian] and you know what that means because you understand Russian, he did not believe that this was a derivative. And in fact, it turns out it is more of a multiplier than a derivative. But still, independent of that, thinking of it as a derivative has helped to generate some results. Term by term differentiation for example, you would have never thought of doing that if you had not been calling it a derivative. So, it is not just a multiplier. The fact that it is not local of course means that it is not just a derivative either.

Do you know some applications of this kind of derivative?

I know that people have talked about them but I do not know much about it myself. Do you remember asking me to write a summary of the applications of the dyadic derivative in Skvortsov's book? Well I never responded because the stuff in Skvortsov's book about dyadic differentiation is actually about BINARY differentiation, an entirely different concept. The binary derivative is usually called the symmetric derivative so you know that dyadic and binary differentiation are entirely two different things, they are not the same. The binary derivative is local, the dyadic derivative is not. So I don't really know any good applications of the dyadic derivative.

Seems that this is a problem for this concept.

Yes, right.

After the first visit to Soviet Union, did you visit it again or not?

I never did, no. What happened was I was going to Hungary, so often, I have been here now seven times, that I was, that was my international travel. And my kids got older it was not so easy to fold up and leave for half a year at a time. And, so I never went back. I have thought about it but I never got around to it. And their situation was a little bit less stable, late eighties, early nineties, when they were making the transition, so there were several reasons why.

And, your cooperation with Hungary is still going on, or you have stopped... Your cooperation with Hungarian group is still active, or not?

So, I am not doing much now. I was an administrator in my University, for nine years, and that just sort of wore me out. And so, I guess the last joint paper I had with them was probably 2003 or 2004. So not, not in the last five years,

Except mathematics, did you have chances to look into other areas in culture of Russia, or Soviet Union?

Well, I have always been fascinated by lots of things. I am fascinated by languages, I actually study philological books as a hobby. I am fascinated by culture. I read a lot about the history of a country before I go there, these are important things... When I was in college I wanted to either be a musician, a historian, a linguist, or a mathematician. And I went into mathematics because I did not have to write so much, because I was not such a good writer. But then it turns out in math, I ended up writing a lot. So, I learned how to write later, but I did not know how to write very well in college.

You have had other international experience, no? Tell us about that, please.

Yes, I've been to 23 countries on five different continents. Some I only went to for a week. For example, I went to one of Butzer's conferences at Aachen (Germany) in 1983 and one of yours at Kupari (Yugoslavia) in 1989.

But if I'm going to stay there more than a week, I always study the culture and history of the host country so I can relate to my new colleagues and I try to write a joint paper as a kind of souvenir of my stay. I was a Fulbright professor to the Indian Statistical Institute (ISI) in Bangalore Spring semester, 1984. I met Siddiqi while I was there but this did not result in a joint publication. However my lectures at ISI formed the basis for Chapter 7 in our book with Schipp and Simon.

My visit to Japan (1996) came after I had published a joint paper with Yoneda and another with Tateoka. I came to know of Yoneda when he sent me some manuscripts of his early work which he was having trouble getting published. As Pichler can tell you, publishing Walsh results was not so easy in these days.

I showed him how to present his work, helped him write up one paper and generalized one of his results from Walsh series to p-series and published that one jointly. Later (1989) he got a prestigious Japanese fellowship (roughly equivalent to our Sloan fellowship - he had a significant salary for a year to go and work with anyone in the world). He chose to come to the University of Tennessee and we had a very nice year talking to one another about Walsh series. Later in that decade, Tateoka got the same fellowship, also came to Tennessee and that's when we worked on the joint paper about Besov spaces which was published in 1995 or 1996.

Another way I experience the local culture if I stay longer than a week is that I start going to church. You probably remember that I am a believer, in fact I'm an ordained Christian minister, and I like to find out how the church is doing wherever I live. That was especially interesting when I stayed in Russia. I find that despite the cultural differences, Christians are much the same all over - they serve the same Lord and have the same perspective about how to live a good life that honors Christ. I guess that some of my openness about these things was startling in Russia and to a lesser extent in Hungary, before 1990, but nowadays it wouldn't be. It might have been more prudent to have hidden it, but the Christian faith affects the way I live life - it's an integral part of me, of who I am.

What do you think about the present status of research in harmonic analysis all over the world, maybe in USA or maybe in Europe or Soviet Union?

I am probably not qualified to speak about that. Just looking at it from a distance, it seems that it is not as energetic as it once was. And you know, all those connections with differential equations that spurred the development of trigonometric series for a while and the connections with martingales that spurred Walsh series have been developed by a lot of people. Some of what I see now is a little repetitive, you know, small improvements and generalizing a bit here and there. And so I don't know, you know, how vigorous the field is right now It seems not as vigorous as it once was. But that is just an amateur's opinion because I am not working right now.

Yes, but what about wavelets theory and related new developments?

Walsh series. Well that's a whole different situation. Much progress is being made on Vilenkin systems and some new and startling results have appeared in the last few years. And as you mentioned, wavelets is a whole new area to investigate that's barely 20 years old now. Although with some wavelets results, I say, my goodness, that looks awfully familiar, you know like the Haar case which I also worked on, which were in some senses the mother of all wavelets. But of course, I understand that the issues are different there and I know that they are looking at different things than we did. But the local character of wavelets plays an essential role just as it did for Haar series. In fact, that's why uniqueness for summable Haar series was solved first. The ability to localize Haar series played a pivotal role in the proof. Yes, there are a lot of new results coming out and the "42" section of Math Reviews increases roughly by 25 more citations every year, e.g., roughly 750 in 2000 and 1000 in 2009.

In one of the publications by doctor Gibbs, with dyadic derivatives, it was written that it was a historic paradox that harmonic analysis developed first on the real line, than to the dyadic group, because it is somehow more natural thing. What do you think about? Is it dyadic shift and dyadic scale more natural or not, or how do you feel about? It was this book by Professor Harmuth who discussed all these issues from engineering point of view.

I am sorry, I just did not look at this kind of application.

I understand, you did not think that way. I am trying to learn about this because it seems that for the dyadic shift and the dyadic group, we cannot find good examples in nature, but still, they look more natural, the addition modulo 2, which is very interesting to discuss. Instead of a circular shift or a linear shift and all these things. And, you mentioned Haar functions. How do you like them?

Well, I personally like them too. You know they are unbounded, so it is a little harder to work with them, but they are very local, their support is small for big indices, like wavelets, and so that makes some things easier. In fact, the technique Talalyan pioneered is that if you can get a uniqueness result for Haar series, then you can obtain a Walsh analogue (for different growth conditions). This was much more tuned to the dyadic case. The older techniques, like Crittenden and Shapiro employed, used the first formal integral of the series which required some subtle topology (perfect sets) and could never have obtained the kind of general results that Skortsov obtained in the 1970's and 80's.

You mentioned today some applications in signal processing by your colleagues. So, do you feel that it is really promising to use these kinds of functions?

Yes, again, here is an area which mushroomed for a while but now is quieting down. One of the biggest motivations for using Walsh functions for signal processing was you could put programs computing the Walsh transform in such tiny space and simulate the Walsh transform on hardware using an open switch for 1 and closed switch for zero (or vice versa). Well, now things have gotten so miniaturized, that's no longer an important consideration. One of the biggest impediments to getting engineers to use the dyadic analysis is that they had no physics behind it. Whereas with the classic Fourier series, they knew, physically, what that was. So, they always wanted to use that, and they used this other, the dyadic stuff, only grudgingly, because they had to save weight and space in putting together this hardware that did these things, like guidance systems, like pattern recognition, like filtering. And the minute that everything got so miniature with space no longer a problem, the engineers went back to the trigonometric transform which they understood intuitively. So that is what happened to the impetus behind applications of Walsh series.

However, there are some applications that are still tailor-made for Walsh functions, and those are applications involving things that jump a lot, like quantum physics, and then there are these genetic algorithms, you know about them? So, a genetic algorithm is a way of optimizing functions that are not differentiate. And the Walsh functions are tailored for that because they jump all around and so people are using them there. These are the two places where applications are still very vigorous.

But Haar functions were the basic wavelets, and then the theory was further developed.

And of course, there are other areas, but I just don't know what they are doing. I'm not keeping up with it. Yes, there are lots of things coming out in wavelets, definitely.

If for instance you had some new student who was eager to learn, would Walsh analysis be a subject that you would suggest to him?

I do not know. If he were very bright, yes.
You know what has happened is we have generated a lot of research over these 42 or 43 years of this business. We asked a lot of questions and we answered them, but the ones that are still unanswered are very difficult. For example, we don't know whether the Walsh-Fourier series of an L log L function converges almost everywhere (it probably doesn't) and we still don't know if the Cantor middle thirds set is a set of uniqueness (it probably is). We need some new tools, and I do not know who is going to come along and discover them. Yes, we need new techniques.

You already said a bit about why Walsh functions were interesting at that time focusing on the easiness of computation. Do you possibly know what the targeted applications were?

I have heard that the Walsh system was used on several Mariner spacecrafts to help with orientation and navigation in space. These spacecrafts were unmanned explorations of our planetary system. I also guess, from knowing who worked on them but the actual information is probably still classified, that some guidance systems on early missiles, especially some anti ballistics missiles, were designed using Walsh transforms.

I know that some astronomers also used the Walsh transform for gathering data. They have a masking procedure for collecting light from the night sky that simulates the Walsh transform. They can then recapture what is actually there by applying the Inverse Walsh transform to this gathered data. As I understand it, one of the problems of trying to view weak stars is that there is not enough light to get very much data. By using masks to gather light in the vicinity of a weak star, they are greatly multiplying the amount of light they collect because of the light from nearby stronger stars. Nevertheless, the effect of that weak star is still being recorded and when they apply the Inverse Walsh transform, the weak star pops out with much greater fidelity than it would otherwise. The use of this technique is now probably less prevalent because with observations being done from space, for example the Hubble telescope, weak stars' light are no longer diluted by our atmosphere, hence much easier to view.

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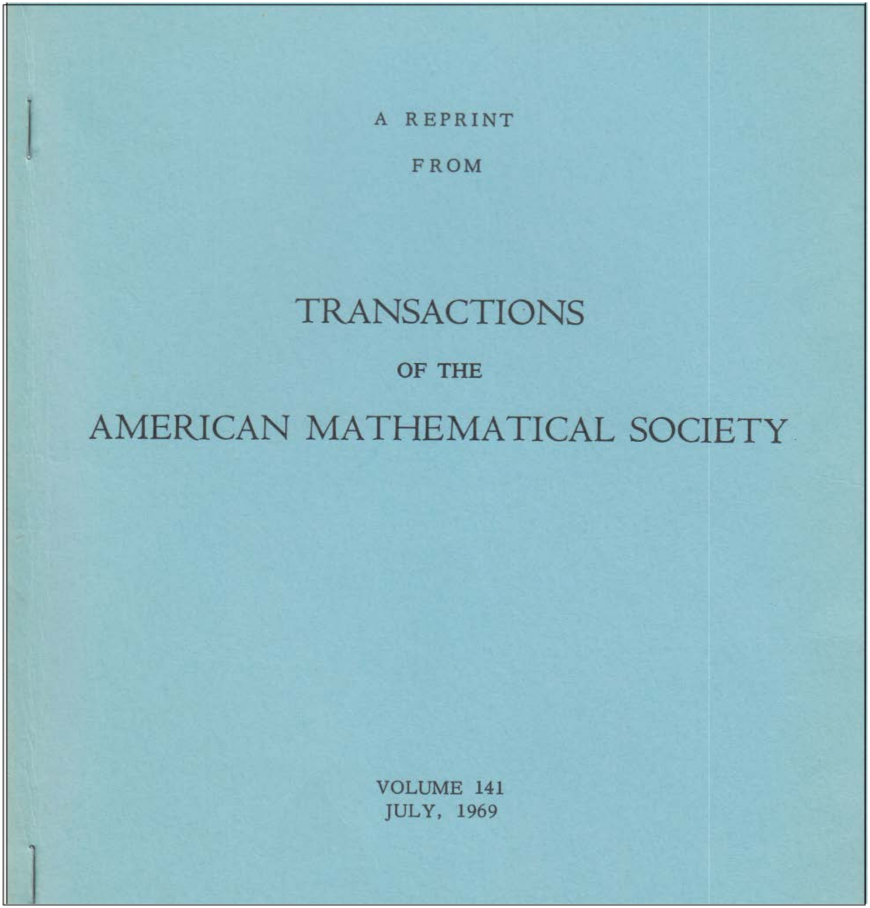

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A UNIQUENESS THEOREM FOR HAAR AND WALSH SERIES(¹)

BY

WILLIAM R. WADE

1. **Introduction.** It is well known that if a trigonometric series converges to an integrable function, except possibly in a countable set, and the series' coefficients converge to zero then that series is the Fourier series of the limit function [8, p. 329].

This problem for Walsh series was open for many years, but in 1965 two independent solutions were published: [2] and [3]. By combining the American and Soviet techniques we are able to obtain a theorem which contains the Walsh series result in [2] but which has a form similar to the theorem in [3]. Following the Soviet example, we will prove the result for Haar series and obtain the Walsh series result as a corollary.

In this paper E will represent a countable subset of [0, 1], and D.R. will represent the set of dyadic rationals. Given a Borel set A, $I_A(x)$ will denote the characteristic function of the set A.

The Haar system $\{\chi_k\}_{k=0}^{\infty}$ is defined as $\chi_0(x)=1$, $\chi_1(x)=I_{[0,1/2)}(x)-I_{(1/2,1]}(x)$; in general writing $k'=2^n+k$, $0 \leqq k<2^n$ where n is the largest power of 2 which is less than or equal to k', we define

$$\begin{aligned} \chi_k(x) = \chi_n^{(k)}(x) &= (2^n)^{1/2} \quad & 2k-2/2^{n+1} < x < 2k-1/2^{n+1}, \\ &= -(2^n)^{1/2} & 2k-1/2^{n+1} < x < 2k/2^{n+1}, \\ &= (2^n/4)^{1/2} & x = k-1/2^n, \\ &= -(2^n/4)^{1/2} & x = k/2^n, \\ &= 0 & \text{otherwise.} \end{aligned}$$

We will denote

$$(2k-2/2^{n+1},\, 2k-1/2^{n+1}) = \Delta_{k'}^{(1)}, \qquad (2k-1/2^{n+1},\, 2k/2^{n+1}) = \Delta_{k'}^{(2)}$$

and these two open intervals will be referred to as the positive (respectively negative) support of the kth Haar function. Alexits [1] proves this sequence is a complete orthonormal system.

We define the Walsh system $\{\Psi_k\}_{k=0}^{\infty}$ by letting $\Phi_0(x)=I_{[0,1/2)}(x)-I_{[1/2,1]}(x)$, $\Phi_n(x)=\Phi_0(2^n x)$ (where Φ_0 is extended by periodicity of period 1) and then defining $\Psi_0(x)=1$, $\Psi_n(x)=\Phi_{n_1}(x)\cdots\Phi_{n_r}(x)$ where $n=\sum_{i=1}^{r} 2^{n_i}$ and the n_i are uniquely

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(¹) This research was done in partial satisfaction for the Doctor of Philosophy degree at the University of California, Riverside, under the direction of Professor Victor L. Shapiro.

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Hence by (1), (28) and Theorem 1,

$$\begin{aligned} a_k &= \sum_{i=2^n}^{2^{n+1}-1} \frac{\varepsilon_{ik}}{(2^n)^{1/2}} \int_0^1 g(x)\chi_i(x)\,dx \\ &= \int_0^1 g(x) \sum_{i=2^n}^{2^{n+1}-1} \frac{\varepsilon_{ik}\chi_i(x)}{(2^n)^{1/2}}\,dx \\ &= \int_0^1 g(x)\Psi_k(x)\,dx \end{aligned}$$

which means that S is the Walsh Fourier series of the integrable function g.

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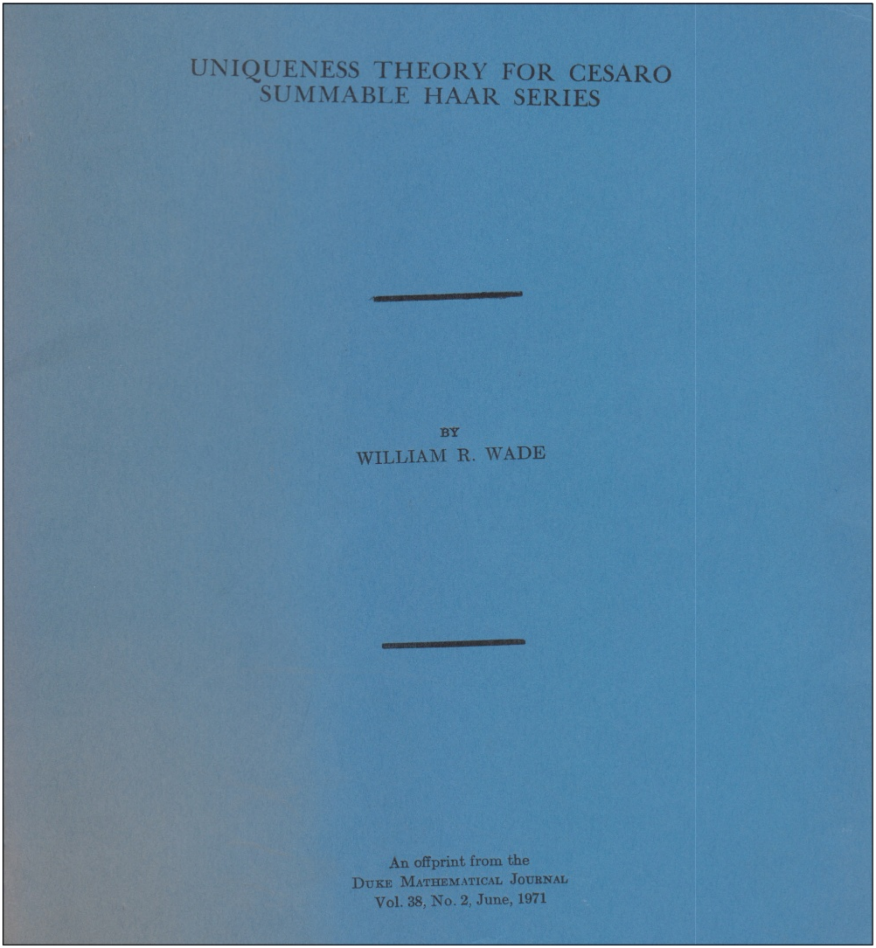

UNIQUENESS THEORY FOR CESARO SUMMABLE HAAR SERIES

BY

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UNIQUENESS THEORY FOR CESARO SUMMABLE HAAR SERIES

By William R. Wade

1. Introduction. Much progress has been made in abstract Fourier analysis by proving analogues of well known trigonometric trsults. The purpose of this research was to investigate the Haar series analogue of a theorem due to Marcel Riesz [4]. Specifically:

Theorem 1. *Suppose the Haar series*

$$S(x) = \sum_{k=0}^{\infty} a_k \chi_k(x)$$

with $a_k = o(k^{\frac{1}{2}})$ *is Cesaro summable to a function* $f(x)$ *which is integrable and finite valued over* $[0, 1]$. *Then* S *is the Haar Fourier series of* $f(x)$.

This theorem is a corollary to Theorem 2.

We define the Haar functions by setting $\chi_0(x) \equiv 1$, $\chi_1(x) = 1$ if $0 \leq x < \frac{1}{2}$ and $\chi_1(x) = -1$ if $\frac{1}{2} < x \leq 1$ with $\chi_1(\frac{1}{2}) = 0$. For any integer $n > 1$ we write it uniquely as $n = 2^m + k$ where $0 \leq k < 2^m$, and as in [5] we define the intervals

(1)
$$\Delta(1, n) = (k/2^m, (k + \tfrac{1}{2})/2^m)$$
$$\Delta(2, n) = ((k + \tfrac{1}{2})/2^m, (k + 1)/2^m).$$

Then the nth Haar function is defined as

$$\chi_n(x) = \begin{cases} \sqrt{2^m} & \text{if } x \in \Delta(1, n) \\ -\sqrt{2^m} & \text{if } x \in \Delta(2, n) \\ -\sqrt{2^m}/2 & \text{if } x = k + 1/2^m \\ \sqrt{2^m}/2 & \text{if } x = k/2^m \\ 0 & \text{otherwise} \end{cases}$$

$\Delta^*(i, n)$ will represent the closure of $\Delta(i, n)$.

Given a Haar series

(2)
$$S(x) = \lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} \sum_{k=0}^{n-1} a_k \chi_k(x)$$

we define its Cesaro sum to be

(3)
$$\sigma(S, x) = \lim_{n\to\infty} \sigma_n(S, x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} S_k(x).$$

The series (2) is said to satisfy *condition G* if for every $x_0 \in [0, 1]$,

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and an interval $\Delta(i_{p_k}, p_k)$ such that

$$z_k \notin \Delta^*(i_{p_k}, p_k) \subset \Delta(i_{p_{k-1}}, p_{k-1}), \qquad k = 1, 2, \cdots \tag{27}$$

$$|\sigma_{n_{i_k}}(S, x)| > k \quad \text{for} \quad x \in \Delta(i_{p_k}, p_k), \qquad k = 1, 2, \cdots$$

and p_k, $\Delta(i_{p_k}, p_k)$ satisfies (6).

Since $\Delta^*(i_{p_k}, p_k) \subset \Delta(i_{p_{k-1}}, p_{k-1})$, this sequence of intervals has a non-empty intersection; let $\xi \in \bigcap_{k=1}^{\infty} \Delta(i_{p_k}, p_k)$. By (27)

$$\limsup_{k\to\infty} |\sigma_{n_{i_k}}(S, \xi)| = \infty,$$

so $\xi = z_q$ for some q. But then $\xi \notin \Delta(i_{p_q}, p_q)$ which contradicts the choice of ξ. Hence $c_k = a_k$ for all k; i.e., S is the Haar Fourier series of f.

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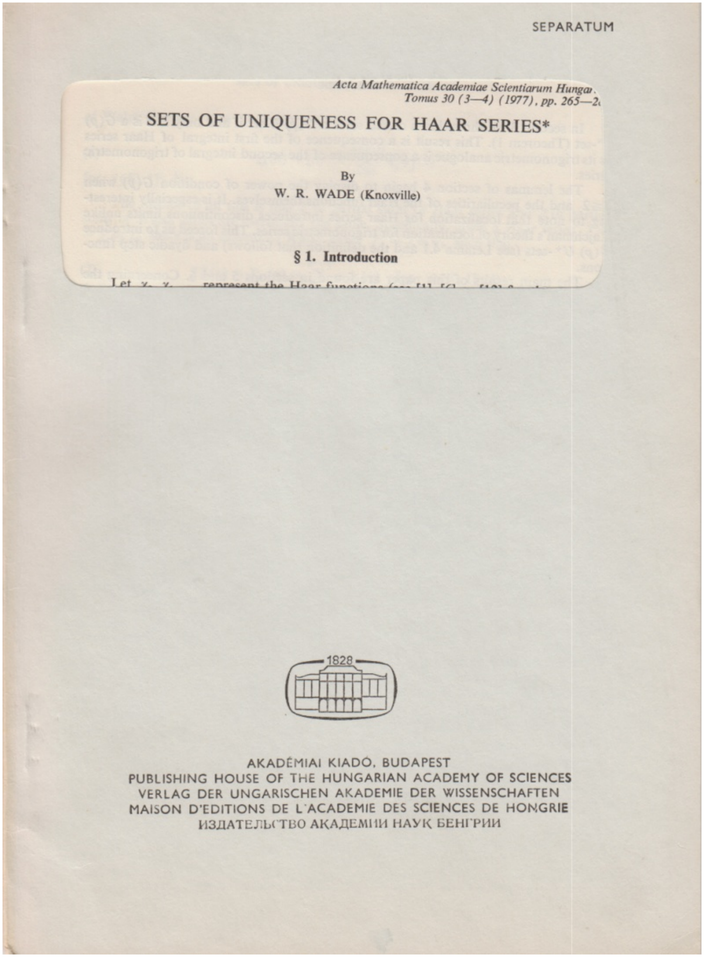

SEPARATUM

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SETS OF UNIQUENESS FOR HAAR SERIES*

By
W. R. WADE (Knoxville)

§ 1. Introduction

AKADÉMIAI KIADÓ, BUDAPEST
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Tomus 30 (3—4) (1977), pp. 265—281

SETS OF UNIQUENESS FOR HAAR SERIES*

By
W. R. WADE (Knoxville)

§ 1. Introduction

Let $\chi_0, \chi_1, \dots$ represent the Haar functions (see [1], [6] or [12] for the standard definition). A set $E \subseteq [0, 1]$ is called a *set of uniqueness* for Haar series if the only sequence $a_0, a_1, \dots$ of real numbers which satisfy

(1) $$\lim_{n\to\infty} \sum_{k=0}^{n} a_k \chi_k(x) = 0 \quad \text{for} \quad x \in [0, 1] \setminus E$$

is the sequence

(2) $$a_k = 0 \quad \text{for} \quad k = 0, 1, \dots.$$

It is known that $E = \emptyset$ is the *only* set of uniqueness for Haar series [7; p. 626]. Hence any further study of sets of uniqueness must be carried out for some restricted class of Haar series. Since Haar functions are defined in terms of square roots of powers of 2, we introduce the following restriction:

Let p be a finite real number. A Haar series $T(x) = \sum_{k=0}^{\infty} a_k \chi_k(x)$ satisfies *condition* $G(p)$ if

(3) $$a_k = o([k]^{(p-1)/2}) \quad \text{as} \quad k \to \infty,$$

where $[k]$ represents the largest power of 2 in k; i.e., $[k] = 2^n$ if and only if $2^n \leq k < 2^{n+1}$. We shall call a set $E \subseteq [0, 1]$ a $G(p)$ *U-set* if the only sequence $a_0, a_1, \dots$ of real numbers satisfying (1) *and* (3) is the sequence (2).

Let $p > q$. Then every Haar series which satisfies condition $G(q)$ also satisfies condition $G(p)$. Hence every $G(p)$ U-set is a $G(q)$ U-set. In particular, $E = \emptyset$ is always a $G(p)$ U-set. On the other hand, since the examples of McLaughlin and Price [7] satisfy condition $G(p)$ for each $p > 2$ we conclude that the empty set is the only $G(p)$ U-set when $p > 2$.

$G(2)$ U-sets have also been characterized. Indeed, in [8] it is shown that a Borel set is a $G(2)$ U-set if and only if it is countable.

The only other known result about $G(p)$ U-sets is also due to Mushegjan [8]. He has constructed a perfect $G(0)$ U-set of measure zero.

A brief outline of the remainder of this paper follows.

In section 2 we recall that Walsh functions and Haar functions are linear combinations of each other, and use that fact to move the formal product theory of Walsh functions over to Haar functions. This is especially interesting since the product of two Haar functions is not, in general, a Haar function.

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We close the paper by noting that the restriction $p \leqq 0$ was crucial to Lemma 2.4. Indeed, if Lemma 2.4. held for any $p_0 > 0$ then the proof of Theorem 4 would establish the existence of a $G(p_0/2)$ U-set with positive measure. This is impossible by Theorem 2.

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When and how did you start your research work in the area of Walsh-Fourier analysis ?

I graduated at Eötvös L. University and earned a diploma as a mathematics and physics teacher. After graduation in 1962 I stayed at the University and started working there as an assistant. At that time I established a closer professional connection with an older colleague of mine, László Pál, who had been an aspirant of Frigyes Riesz. Thanks to him we learned about the spiritual climate that had dominated the Mathematical Institute during the 1950's and about the stories concerning the true friendship between Frigyes Riesz and Lipót Fejér, and also the anecdotes about them. When I started my university studies in 1957, Riesz was already dead and Fejér did not give analysis lectures anymore. However, I once saw Fejér at the University.

Due to Fejér, Riesz and Alfréd Haar, there is still an important tradition of the theory of Fourier-series, approximation theory and functional analysis. This heritage was continued by György Alexits and Pál Turán, in Budapest, and Béla Szõkefalvi-Nagy and Károly Tandori, in Szeged, who established the domestic schools of approximation theory and functional analysis. Unfortunately, they are all dead by now.

It was that environment where I started my work with Fourier-series. I first heard about the Walsh system in connection with a problem of László Pál. He worked on a problem of Hardy on the convergence of products of multiplication of hyper harmonic series. The convergence with probability one of them led to the question of convergence of two-dimensional Rademacher-series, which are Walsh-series of a special type. We found Walsh's paper in the library of the University of Szeged.

The original definition of Walsh was too complicated to solve this problem. After having found the basic paper of N.J. Fine on Walsh-series, and applying the explicit form of the Lebesgue constants for the Walsh system, Pál solved the problem.

Pál, L.G., "On general multiplication of infinite series", *Acta Sci. Math.* (*Szeged*), **29,** 1968, 317–330.
(http://www.ams.org/mathscinet/search/journaldoc.html?cn=Acta_Sci_Math_Szeged)

Fine showed that the seemingly artificial Walsh system can be considered as the trigonometric system (the character system) of the Cantor (dyadic) group. This work made a great impression on me, since this new concept allowed us to handle the problems with the tools of harmonic analysis. I did not know about the paper of N.Ja. Vilenkin at that time, in which this idea had already appeared in a much more general form. The character systems of those groups were named after him.

In my first paper, which was the basis of my university doctor dissertation, I constructed a continuous function whose Walsh-series diverges at a point.

Schipp. F., "Construction of a continuous function whose Walsh series diverges at a prescribed point", *Annales Universitatis Scientiarum Budapestinensis de Roland Eötvös Nominatae*, Sectio Mathematics, Tomus iX, 1966, 103-108.

Later I constructed another very simple counter example, which can be considered as the dyadic analogue of the Fejér construction for divergent trigonometric Fourier series. This construction was further developed by Peter Simon by giving several examples for divergent Vilenkin series.

Simon, P., "On the divergence of Vilenkin-Fourier series", *Acta Math. Hungar.*, 41 (1983), No. 3-4, 359–370.

Tell me about your impressions on the first papers that you read in this area?

Another fundamental paper of this subject is due to R.E.A.C. Paley. He introduced a new and very useful enumeration of the Walsh system. Later this paper had a great influence on the development of martingale theory.

At that time in Hungary, thanks to mainly the works of György Alexits and Károly Tandori, many people were dealing with the convergence and summability problems of general orthogonal series. The monograph of Alexits was published at that time, and it contained the famous deep and subtle divergence constructions of Tandori. In the Soviet Union, the Russian translation of the book of Kaczmarz and Steinhaus, with an appendix on Vilenkin systems written by P.L. Uljanov, came out. These works influenced my scientific activities in a great deal.

Here I would like to mention our connection with the Polish mathematicians which was initiated by László Pál. My first official visit was to Poznan, Poland in 1964. It was a great experience to me to meet Professor W. Orlicz, the head of the Department of Mathematics and Professor Z. Ciesielski, who was then writing his important papers on Franklin and Schauder systems. This relationship deepened during mutual visits and the international conferences organized at the Banach Centre. It was then when I first heard about the massacre in Katyn. Later I learned that, J. Marczinkiewicz, whose result I respected very much had been one of the victims there. We can only guess the great loss that his death caused to the theory of Fourier series.

ANNALES

UNIVERSITATIS SCIENTIARUM BUDAPESTINENSIS DE ROLANDO EÖTVÖS NOMINATAE

SEPARATUM

SECTIO MATHEMATICA

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1966

CONSTRUCTION OF A CONTINUOUS FUNCTION, WHOSE WALSH' SERIES DIVERGES AT A PRESCRIBED POINT

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Let the functions $r_n(x)$, $(n = 0, 1, \ldots)$ be defined by

$$r_0(x) = \begin{cases} 1, \text{ if } x \in \left[0, \frac{1}{2}\right), \\ -1, \text{ if } x \in \left[\frac{1}{2}, 1\right), \end{cases}$$

$$(1) \qquad \begin{aligned} r_0(x+1) &= r_0(x), \\ r_n(x) &= r_0(2^n x) \quad (n = 1, 2, \ldots). \end{aligned}$$

This is the well known Rademacher-system, but the definition of $r_0(x)$ for $x = \frac{1}{2}$ differs from that of the original one.

Let the functions $\psi_n(x)$ be defined by

$$(2) \qquad \begin{aligned} \psi_0(x) &\equiv 1, \\ \psi_n(x) &= r_{\nu_1}(x) \cdot r_{\nu_2}(x) \ldots r_{\nu_s}(x), \end{aligned}$$

$$\text{for} \quad n = 2^{\nu_1} + 2^{\nu_2} + \ldots + 2^{\nu_s} \quad (\nu_1 > \nu_2 > \ldots > \nu_s \geqq 0).$$

This is the Walsh' system, which is known as complete and ortonormalized.

The trigonometric- and the Walsh'-system has a lot of properties similar. Thus for example the Lebesgue' constants are unbounded for both of these systems. One can deduce from this fact – by known general results – the existence of a continuous function, with Walsh' Fourier series diverging at prescribed point [1], [2].

In the following we give a construction, which leads to a continuous function $F(x)$ with the property mentioned above. To our knowledge such a construction has not been known yet, though for the corresponding function of the trigonometric system a lot of different constructions by various authors can be found (cf. e.g. [5], pp. 50–51.), [3], [4].

Now, considering that

$$\int_{1/2^t}^{1/2^{t-1}} f_{2n^4}(u)\,du = \frac{1}{4}\left(\frac{1}{2^{t-1}} - \frac{1}{2^t}\right) = \frac{1}{2^{t+2}}$$

holds for every natural number t with $t \leq 2n^4$, and taking into account the definition of the p_n' s, (17) results that

$$\text{(18)} \qquad A_n = \frac{1}{n^2} \sum_{j=(n-1)^4}^{n^4-1} \left\{ 2^{2j} \sum_{t=2n^4}^{2j+2} \int_{1/2^t}^{1/2^{t-1}} p_n(u) f_{2n^4}(u)\,du + 2^{2j} \frac{1}{2^{2j+3}} \right\} =$$

$$= \frac{1}{n^2} \sum_{j=(n-1)^4}^{n^4-1} \left\{ 2^{2j} \sum_{t=2n^4}^{2j+2} (-1)^t \int_{1/2^t}^{1/2^{t-1}} f_{2n^4}(u)\,du + \frac{1}{8} \right\} =$$

$$= \frac{1}{n^2} \sum_{j=(n-1)^4}^{n^4-1} \left\{ 2^{2j} \sum_{t=2n^4}^{2j+2} (-1)^t \frac{1}{2^{t+2}} + \frac{1}{8} \right\}.$$

From (18) considering the non-negativity of the inner sums, it follows, that

$$\text{(19)} \qquad A_n > \frac{1}{8n^2} \sum_{j=(n-1)^4}^{n^4-1} 1 = \lambda_n,$$

where $\lambda_n \to +\infty$ when $n \to \infty$.

Thus by (15), (16) and (19) the statement (13) holds for the sequence $S_{m_n}(F; 0)$. Qu. e. d.

We remark — as well known — that

$$S_{2^n}(F; 0) \to F(0) \quad (n \to \infty),$$

so the sequence $S_n(F; 0)$ does not converge even in the weaker sense ($S_n(F; 0) \to \infty$).

Using a method of Fine, one can obtain from this construction a continuous function, whose Walsh—Fourier series diverges at an arbitrary prescribed point.

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After having the university doctor degree I continued my research work in order to earn the so called candidate degree in mathematics. My supervisor was Károly Tandori. He continued, significantly improved and in a certain sense finalized the results of professor Menchoff, the leader of the Moscow school of real functions theory. As a consequence he reached great reputation in Moscow which reflected also on me and Ferenc Móricz, his students at that time. Alexits and Nikolskii have initiated a series of international conferences on approximation theory and Fourier series. Later, besides Hungary and the Soviet Union also Bulgaria and Poland joined the program, and colleagues over there organized successful conferences. These conferences provided good opportunities to keep in contact not only with the researchers from socialist countries but also with western researchers. An important point in the collaboration was the foundation of the journal *Analysis Mathematica* jointly by the Soviet and the Hungarian Academies.

The reputation of the journal was well established by the first chief editors, professors Nikolskii and Szőkefalvi-Nagy. Throughout the years *Analysis Mathematica* became a worldwide acknowledged periodical in the theory of Fourier series including dyadic analysis. I felt honored when I became a member of the editorial board. Professor Nikolskii at age 105 is still the chief editor from the Russian side.

Did you have a chance to travel to the former Soviet Union *at that time?*

Within my research program toward the candidate degree there were opportunities for travel to research centers abroad. That is how I went to the Soviet Union. Later I visited Moscow several times in connection with *Analysis Mathematica* and the cooperation between Eötvös L. University (ELTE) and the Moscow State University (MGU).

On my first visit to Moscow I talked about my result in the seminar of Menchoff. Since these were related to the Walsh functions I tell some more about them. Walsh constructed the orthonormal system, which was later named after him, on the basis of some properties, like parity and the number of changes of sign, of the trigonometric system. In his construction he used complicated recursion formulas. He named the terms corresponding to *sine* and *cosine* as *sal* and *cal.* This original system is widely used in technical applications. In his fundamental paper mentioned above, Paley generated the Walsh functions as products of Rademacher functions. This so called Paley ordering is the one used by mathematicians in most cases. The theoretical results are mainly for this ordering. There exists however a third version, called Kaczmarcz ordering, which is in strong connection with Hadamard matrices.

At the Menchoff seminar I showed that these rearrangements can be described by linear bit transformations over the Galois field with element 2. Moreover, they generate a measure preserving map of the variable. The rearrangements in question can be originated from the Walsh-Paley system by such transformations of the variable. By means of this observation most of the results known for the Walsh-Paley system can automatically be transferred to the other two systems as well. I published these results in the journal *Mat. Zametki.*

МАТЕМАТИЧЕСКИЕ ЗАМЕТКИ

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О НЕКОТОРЫХ ПЕРЕСТАНОВКАХ РЯДОВ ПО СИСТЕМЕ УОЛША

Ф. Шипп

В работе рассматривается сходимость некоторых перестановок рядов по системе Уолша — Пэли. В частности, показано, что системы Уолша и Уолша — Качмажа являются системами сходимости. Библ. 6 назв.

1. В большинстве работ, посвященных рядам Уолша, исследуются ряды по так называемой системе Уолша — Пэли [1]. В некоторых работах, а также в техническом применении функций Уолша (см., например, [2], [3]) используется подлинная система Уолша [2]. Эта система является такой перенумерацией системы Уолша — Пэли, при которой число перемен знака n-й функции на интервале $(0, 1)$ равно n. Другой перенумерацией системы Уолша — Пэли является так называемая система Уолша — Качмажа, исследовавшаяся в ряде работ (см., например, [4]).

Отметим, что в применении названий различных типов функций Уолша литература непоследовательна. Например, в [5] подлинную систему Уолша называют системой Уолша — Качмажа. Здесь мы будем следовать названиям, используемым в [4].

В данной работе мы исследуем два общих типа перестановок системы Уолша — Пэли (так называемые линейные и кусочно-линейные перестановки). Эти перестановки содержат в себе в качестве частных случаев системы Уолша и Уолша — Качмажа. Будет показано, что такие перестановки рядов по системе Уолша — Пэли с точки зрения сходимости ведут себя аналогично рядам Уолша — Пэли.

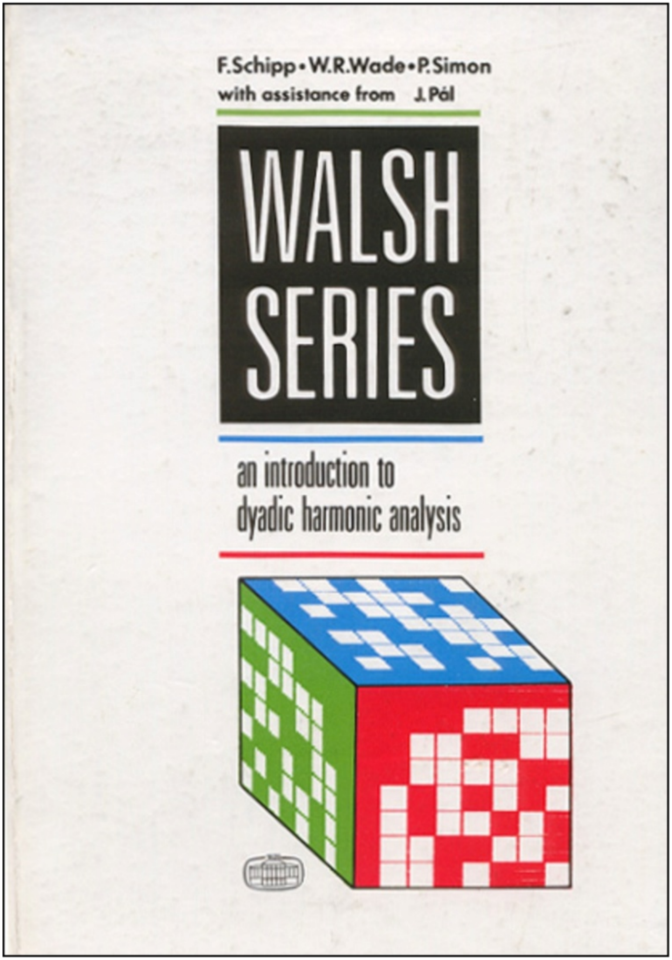
Menchoff, in spite of being rather old that time, followed my talk with vivid interest. He noted that the idea of bit transformation came up in special cases concerning rearrangements of Rademacher functions even in the 30's. It was then when I became acquainted with Nikishin, Bockarev, Golubov, and later with Kashin, the students of Ul'janov. Nikishin, who died very young, deepened the theory of function series in a remarkable degree. The others are still very active and respected researchers of our field. Golubov, who took part of this conference also, studied Haar series with several other Russian mathematicians that time. This was the time when the Haar-system got in the focus of interest after a long silence. The results of their research showed that the Haar-system play a unique role among the orthonormal systems.

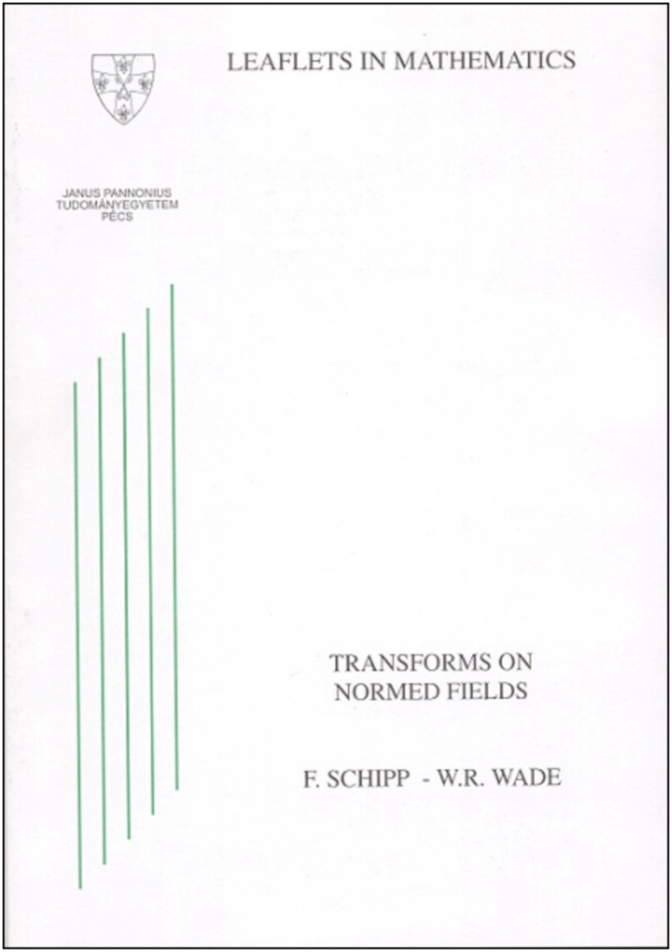
Those days the Soviet Union launched a series of books, *Itogi Nauki*, in which the latest results in mathematics were summarized. In this series there were published the review paper of Balasov and Rubinstein on Walsh series, and the work of Golubov on Haar series. Through these papers we received very useful information about the research activities in the Soviet Union.

Our collaboration with Professor Wade started in 1983. There was a World's Fair on energy in Knoxville, USA in 1982. The minister of industry of Hungary visited the Fair and the University of Tennessee, and looked for possible fields of cooperation. Professor Wade who already knew some of my works at that time proposed collaboration in the theory of Walsh functions. His initiation was backed by the NSF providing support for one visit to Hungary and to the USA each. Then we worked out a plan for a three year long collaboration and handed in an application for financial support. Our application was granted by the Hungarian Academy of Sciences and by the National Science Foundation in the USA. The publication of a monograph on Walsh series was in the center of our efforts. By that time we already had an assignment to write such a book from the Hungarian publisher *Akadémiai Kiadó*. Me and my colleagues Jenõ Pál and Péter Simon were very happy when Professor Wade agreed on helping us as a co-author of the book. His experience in writing review papers and his wide knowledge in the field of dyadic harmonic analysis turned to be very valuable in several respects including the choice of topics, historical comments, etc. As an expert in problems on unicity, which topic was a little far from our interest, he took the whole responsibility concerning those parts of the book. Also his high standard in English added a great value to the book. He had to work, maybe even suffer, a lot to put our manuscripts, although mathematically correct, into readable English.

It took 10 years from the very beginning to the printing of the book. In the NFS grant there was an amount of money allocated for typing expenses. A secretary typed the manuscript by means of an IBM typewriter. It happened more than one time when we had to change the material of a chapter written earlier. Then the whole chapter had to be retyped. By the time we finished with the book TEX came out. Professor Wade and his son Peter were among the first to learn how to use it and they together put the whole book into TEX. Finally this version ready to print was published jointly by Akadémia Kiadó and Adam Hilger in 1990. The book was awarded by the Academy of Sciences by the prize of excellence.

The first grant was followed by a second one. This provided opportunity for several colleagues of mine to travel to the USA and gain experience there. At that time such an opportunity was not as common as it may be nowadays.

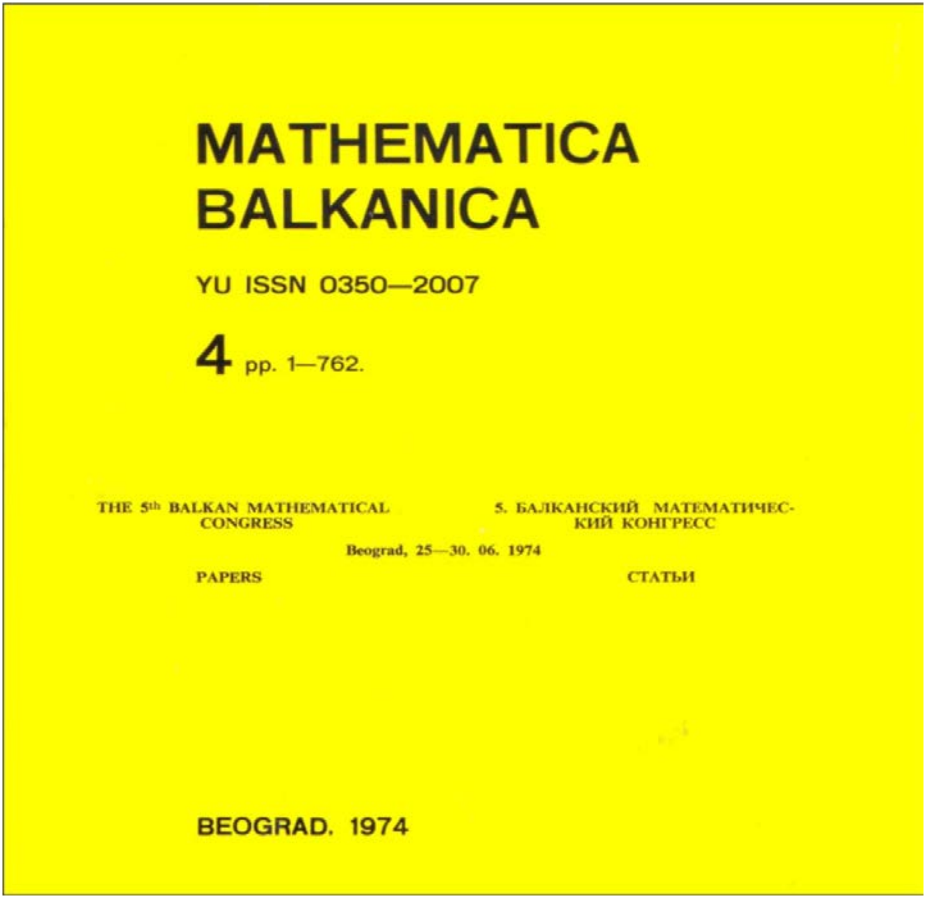

F.Schipp • W.R.Wade • P.Simon
with assistance from J.Pál
WALSH SERIES
an introduction to
dyadic harmonic analysis


LEAFLETS IN MATHEMATICS
JANUS PANNONIUS
TUDOMÁNYEGYETEM
PÉCS
TRANSFORMS ON
NORMED FIELDS
F. SCHIPP - W.R. WADE

During our work the famous monograph of Zygmund served as a model. Some chapters deal with problems similar to those in Zygmund's book. As an example I mention the chapters on dyadic derivative, unicity problems, Walsh-Fourier transform. These include also the contribution of the authors to the corresponding topics.

The discrete version of the dyadic derivative was introduced by Gibbs, the continuous version is due to Butzer and Wagner. The first time I encountered this concept was in a paper in *Analysis Mathematica*. In that paper Butzer and Wagner worked out the foundation of the theory based on the concept of strong derivative. They introduced dyadic integration and showed that that the dyadic derivative plays a role in Walsh analysis that is similar to that of the traditional derivative in trigonometric Fourier analysis. As a problem they posed the question whether the analogue of the theorem of Lebesgue on the almost everywhere differentiability of the integral function holds for the pointwise dyadic derivative. I solved the problem and presented the result in a talk at the *V Balkan Mathematical Congress*. It was published in *Mathematica Balcanica*. Then I got in personal connection with Professor Butzer and his students, and by mail with Dr. Gibbs. Professor Stanković has a great merit in publicity and application of dyadic analysis. He organized workshops in order to bring together scientist of our field. In one of these workshops I was fortunate to meet Dr. Gibbs personally.

The physical interpretation of dyadic derivative is still on the agenda. What we see for sure is that the dyadic derivative and technique based on it can be effectively applied in the investigations concerning the problems of summability of Walsh series. Here I would like to mention the contribution of my colleague Sándor Fridli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THE 5th BALKAN MATHEMATICAL CONGRESS
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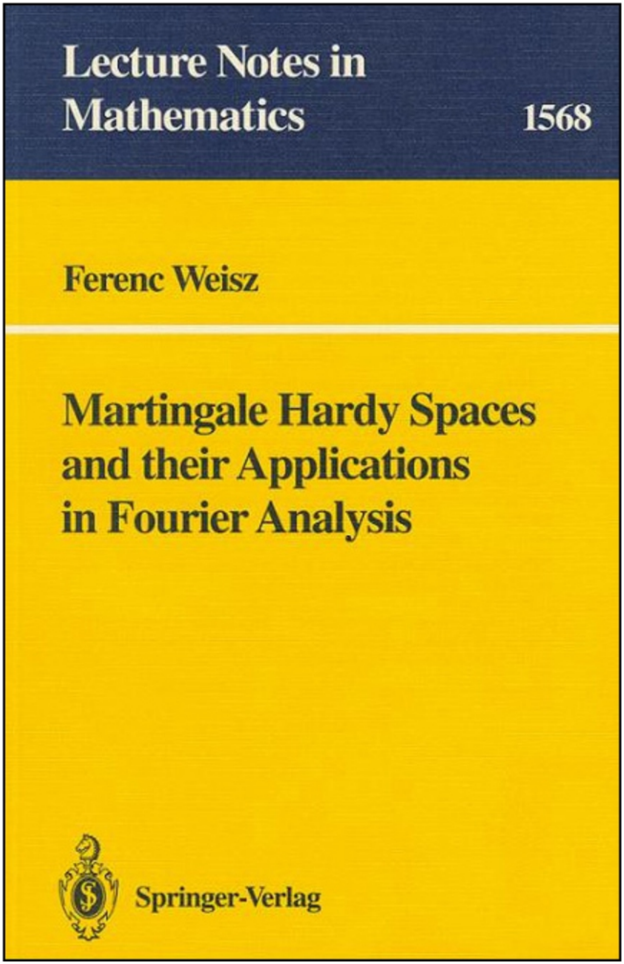
Beograd, 25—30. 06. 1974
PAPERS
СТАТЬИ
BEOGRAD. 1974

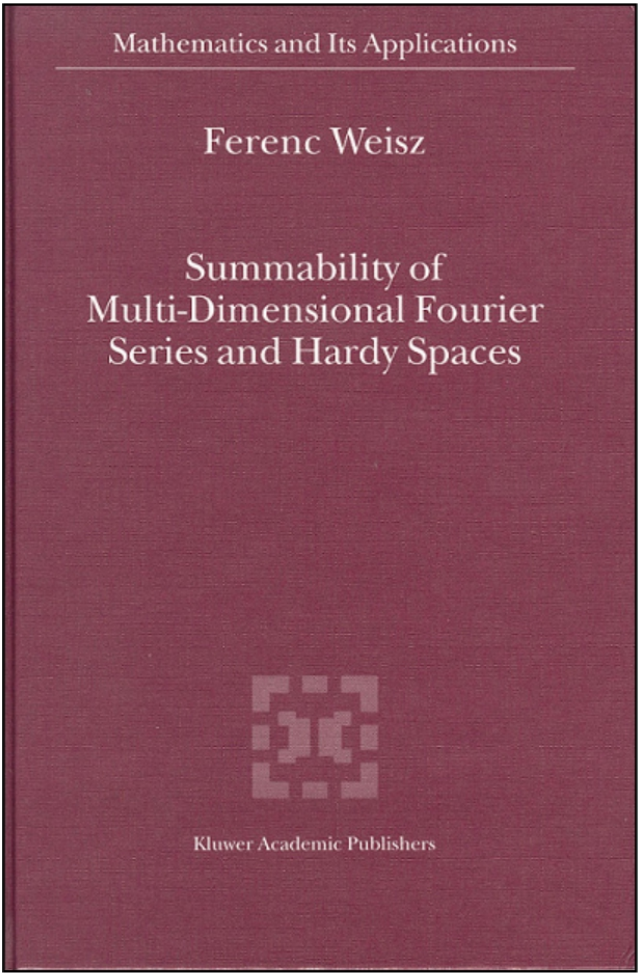
to the development of this area.

Fridli, S., "Hardy spaces generated by an integrability condition", *J. Approx. Theory* **113** (2001), No. 1, 91–109.

The research activity in the areas of multidimensional Walsh series, Vilenkin series and Walsh-Fourier transform is just as intense as it has been. Besides the mentioned chapter of our book two books written by Professor Weisz contain the latest results.

It is especially worth to note the probabilistic approach that has been used in the chapter on almost everywhere convergence of Walsh-Fourier series. Due to Burkholder's works I am convinced that the concepts and the techniques of martingale theory, like martingale transform, martingale maximal theorem and stopping time, can effectively be applied in the investigation of hard problems in dyadic analysis. The partial sums of Haar series are dyadic martingales while the Walsh partial sums form dyadic martingale transforms. Making use of this connection with martingale theory several complicated proofs became simpler and easier to understand. For instance the problem of convergence in L^p can be solved on this base. The power of these methods is clearly demonstrated by the fact that it was successfully applied not only in dyadic analysis but also in connection with some classical inequalities of the theory of trigonometric series. For instance the inequalities of M. Riesz and Kolmogorov on the trigonometric conjugate function can be solved this way with finding also the best constants in them.


Lecture Notes in
Mathematics
1568
Ferenc Weisz
Martingale Hardy Spaces
and their Applications
in Fourier Analysis
Springer-Verlag


Mathematics and Its Applications
Ferenc Weisz
Summability of Multi-Dimensional Fourier Series and Hardy Spaces
Kluwer Academic Publishers

The problem of almost everywhere convergence turned to be much harder than the norm convergence case. This is reflected in the story of Luzin's conjecture, perhaps the most famous problem of Fourier series. It started at the beginning of the last century. Luzin conjectured that the Fourier series of every square integrable function converges almost everywhere. The proof, which was without doubt the greatest achievement in analysis in the last century, was given by Carleson in 1964. It's analogue for the Walsh system was proved by Billard for functions in L^2 , and the general case of L^p (p>1) was proved by Sjölin, the student of Carleson. It was our definite aim that this result should be included in our book. This inspired me to understand Carleson's proof and to put it in another perspective. I have extended the concepts and the results of martingale theory to stochastic processes whose index set (time domain) was not linearly ordered but a binary tree instead. This way the problem of almost everywhere convergence of Walsh series became easier to handle. Moreover the proof can be adapted to a rather wide class of orthogonal systems. These are product systems of martingale differences (see, pages 94,95). It is worth to highlight that these systems have a property which is very important in applications. Namely, the Fourier coefficients with respect to them can be computed with a fast algorithm similar to FFT. This way almost every known one and multidimensional FFT algorithm can be represented in a uniform way.

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Analysis Mathematica, 2 (1976), 49—64

On L^p-norm convergence of series with respect to product systems

F. SCHIPP

To Professor K. Tandori on his 50th birthday

In this paper we concern ourselves with the L^p-norm convergence of sequences of operators, which are in a certain relation with conditional expectation operators. To indicate the character of the results obtained we mention some concepts and notations in advance.

Let $(X, \mathscr{A}, \mu)$ be a probability space, $\mathscr{A}_0=\{X, \emptyset\}\subset\mathscr{A}_1\subset\ldots\subset\mathscr{A}_n\subset\ldots\subset\mathscr{A}$ a sequence of σ-algebras. Denote by E_n the conditional expectation operator relative to $\mathscr{A}_n$, and let $\Delta_0=E_0$, $\Delta_n=E_n-E_{n-1}$ $(n=1, 2, \ldots)$. The facts that $E_0 f$ is equal to the constant function $\int_X f\,d\mu$ and that E_n has properties similar to those of an integral make it possible to extend the concepts connected with the integral to the conditional expectations E_n. For example, as a generalization of boundedness (i.e., (p, q) type) of an operator $T: L^p(X, \mathscr{A}, \mu)\to L^q(X, \mathscr{A}, \mu)$ $(1\leqq p,\ q<\infty)$ we introduce the following concept: the operator $T: L^p(X, \mathscr{A}, \mu)\to L^q(X, \mathscr{A}, \mu)$ is said to be of type $(\mathscr{A}_n, p, q)$, if there exists a number $K>0$ such that for every function $f\in L^p(X, \mathscr{A}, \mu)$

$$\{E_n(|Tf|^q)\}^{1/q} \leqq K\{E_n(|f|^p)\}^{1/p}$$

is satisfied. The infimum of these numbers K is called the $(\mathscr{A}_n, p, q)$-norm of the operator T and is denoted by the symbol $\|T\|_{(\mathscr{A}_n, p, q)}$.

In this note we will show among others that if for a sequence T_n $(n=1, 2, \ldots)$ of operators

$$(\Delta) \qquad \Delta_n\circ T_n = T_n\circ\Delta_n = T_n \quad (n = 1, 2, \ldots)$$

holds, then certain boundedness properties of T_n are inherited by the operator

$$(*) \qquad T = \sum_{n=1}^{\infty} T_n.$$

For example, we have the following theorem: *If the operators T_n satisfy condition (Δ) and are of type $(\mathscr{A}_{n-1}, r, r)$ for $r=2$ and $r=p$ $(2\leqq p)$, uniformly, i.e., there exists*

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Pointwise convergence of expansions with respect to certain product systems

F. SCHIPP

In this paper we apply CARLESON's method [1] to the study of a.e. convergence of series with respect to certain product systems. Among others, we consider systems

$$\Phi=\{\varphi_n:\ n=1, 2, \ldots\}\subset L^1(X, \mathscr{A}, \mu)$$

($(X, \mathscr{A}, \mu)$ is a probability space) such that with some sequence of σ-algebras

$$\mathscr{A}_0=\{X, \emptyset\}\subset\mathscr{A}_1\subset\ldots\subset\mathscr{A}_n\subset\ldots\subset\mathscr{A}$$

the following conditions are satisfied: φ_n is $\mathscr{A}_n$-measurable, $|\varphi_n|=1$ and $E(\varphi_n|\mathscr{A}_{n-1})=0$ $(n=1, 2, \ldots)$.

We will show that the product systems of such systems are convergence systems. Hence it follows that the product system of any system consisting of independent functions with zero expectation and absolute value 1 is a convergence system.

As a special case of this we obtain a result of P. BILLARD [2] that asserts that the Walsh—Paley system is a convergence system. We notice that conversely our theorem in the real case can be easily deduced from this theorem of Billard (see, e.g., [3], [4]). But in the complex case the methods used in [3] and [4] can not be applied.

From Theorem 1 it follows easily that the Walsh system and the trigonometric system in Kaczmarz's rearrangement are also convergence systems.

The results mentioned above are simple consequences of Theorem 2 proved in the present paper. In this theorem the operator

$$S^*f=\sup_\alpha|S_\alpha(f\bar{\psi}_\alpha)|$$

is considered. This operator is defined by certain operators S_α, where ψ_α are elements of the product system of certain systems Φ_n $(n=1, 2, \ldots)$. In an earlier paper [5] we proved that then operators S_α are of type (p, p) $(2\leqq p<\infty)$ uniformly in α. In this paper we will show (Theorem 2) that the operator S^* is of weak type (2, 2).

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22. NUMERICAL METHODS, KESZTHELY (HUNGARY), 1977.

FAST FOURIER TRANSFORM AND CONDITIONAL EXPECTATION

F. SCHIPP

1. INTRODUCTION

Let f be a complex-valued function, defined on the set $\{0,1,2,\dots,M-1\}$ $(M\in N^* := \{1,2,3,\dots\})$ and let $\hat{f}$ denote the discrete trigonometric Fourier transform (the discrete trigonometric Fourier coefficients) of f:

$$(1) \qquad \hat{f}(n) := \sum_{k=0}^{M-1} f(k)\exp(-2\pi\, ikn/M), \quad (n=0,1,\dots,M-1).$$

When computing the values of $\hat{f}$ on the basis of (1), we need $O(M^2)$ operations (additions and multiplications). By applying Fast Fourier Trasform (FFT) algorithms we can achieve the same by using only $O(M \log M)$ operations (see [1], [3]-[11]). As to the history of FFT, see [5]. An analogue of the well-known Cooley-Tukey algorithm can be used for quickly computing the Walsh-Fourier coefficients, too (see the corresponding bibliography of the book [2]).

Matematica Pannonica
13/2 (2002), 265-275

FAST FOURIER TRANSFORM FOR RATIONAL SYSTEMS

Schipp Ferenc (ELTE, SZTAKI, PTE)

Dedicated to Professor Wolfgang Breckner on his 60th birthday

Abstract. Conditionally orthogonal systems of rational functions are introduced and their product systems are examined. It is proved that under certain conditions these product systems form a discrete orthogonal system and the partial sums of their Fourier-series can be obtained in a useful form. Also byorthogonal systems of this type are investigated.

Expansions by orthogonal and biorthogonal systems play an important role in mathematics and in applications. Several methods are available for constructing such systems. In a Hilbert space, for instance, the Gram-Schmidt method transforms a linearly independent system into an orthonormed one. Orthogonal polynomials, the Franklin–system and its generalizations, orthogonal systems consisting of rational functions (discrete Laguerre, Kautz, Malmquist–Takenaka systems) are examples that can be derived this way [2], [3].

In the middle of the 1970's the author introduced a new method for constructing orthogonal systems starting from some conditionally orthogonal functions [5], [6], [7]. Several classical systems, including the trigonometric, the Walsh system, or the Vilenkin system, character systems of additive and multiplicative groups of local fields, UDMD– and Walsh–similar systems, can all be constructed by using this method [10], [11],[13],[15], [16], [17]. One of the key concepts in our construction is the notion of product systems of conditionally orthogonal systems. These systems have important theoretical properties that are useful in numerical computations, too. For instance, Fourier–coefficients and partial sums can be computed by applying fast algorythms similar to FFT [8], [9], [12], [14].

In this paper we introduce conditionally orthogonal systems of rational functions, and examine their product systems. We prove that under certain conditions these product systems form a discrete orthogonal system and the partial sums of their Fourier–series can be obtained in a useful form. We also investigate biorthogonal systems of this type.

Expansion of this type can be used in control theory [2].

Key words and phrases. Rational functions, Blaschke-funktions, product systems, FFT algorithms.
This research was supported by the grant OTKA/T032719

Also the other chapters of our book inspired further research in the corresponding topic. For instance, the results in dyadic Hardy spaces served as starting points for the introduction of sequence Hardy spaces by Sándor Fridli. He used these Hardy spaces to find the best possible Sidon type inequalities in a sense, to improve the previous results on integrability and L^1 convergence of Fourier series, and on Hörmander type multipliers.

Fridli, S., "Hardy spaces generated by an integrability condition", *J. Approx. Theory,* **113** (2001), No. 1, 91–109.

Special emphasis was put on the question on bases in different Banach spaces in our book. The Haar and the Franklin systems play important roles in the solution in many problems concerning bases. Most of the results presented have been transferred to wavelet since then, see e.g. Meyer's books. The importance of the Walsh system was demonstrated also by the solution of the basis problem, one of the significant problems of analysis, given by Enflo. In his proof the Walsh functions are of special importance. In the book we presented another construction by giving an example for a BMO type separable Banach space that has no basis. The inseminating effect of the Walsh system in the various fields of mathematics was summarized in the volume "Joseph L. Walsh, Selected Papers", issued by Springer to celebrate that Walsh was born 100 years ego.

The importance of the technique and approach of dyadic analysis did not lessen by time. It is known that the nowadays intensively applied wavelet transform traces back to the Haar system. This may inspire us to make use all the information that has been accumulated concerning the Haar system in the theory of wavelets. It would be useful for example to find a martingale theoretic interpretation, similar to the case of the Haar system, of wavelets, and so to the open door for the application of the effective technique developed there.

Zbigniew Cieselski, a PhD student of W.R. Orlics at Ph.D. Uniwersytet im. Adama Mickiewicza Poznan, 1960.

Józef Marcinkiewicz was a student of Antoni Zygmund, who also worked with Juliusz Schauder, and Stefan Kaczmarz. Marcinkiewicz was killed in the Katyn massacre.

The **Katyn massacre** or **t**he **Katyn Forest massacre** was a mass execution of Polish citizens considered as nationalists by the Soviet secret police NKVD, in April and May 1940 around the villages Katyn and Gnezdovo about 19km from Smolensk, Rusia. The number of victims is estimated at about 22,000, the most commonly cited number being 21,768.

Sergey Mikhailovich Nikolsky is a Russia mathematician who made essential contributions to Functional analysis, approximation of functions, Quadrature formulas, and other areas of mathematics.

Béla Szőkefalvi-Nagy was a Hungarian mathematician who served as the editor-in-chief of the *Zentralblatt für Mathematik*, the *Acta Scientiarum Mathematicarum*, and *the Analysis Mathematica*.

Dmitrii Evgenevich Menchoff was a Russian mathematician, a student of Nikolai Luzin, renowned by his contributions to trigonometric theory.

Appeared in: Joseph L. Walsh, Selected Papers,
Springer, 1999,pp.129-135, Eds.:T.J.Rivlin and E.B. Saff

ON THE WALSH SYSTEM

F. SCHIPP (Budapest)

INTRODUCTION

A number of major fields of modern mathematics developed in the first decades of this century. Among them are measure theory, functional analysis, and real analysis. That was the time for the mathematical foundation of probability theory and a new age began in the theory of Fourier series. There are three orthogonal systems which played a fundamental role in this process. One of them was introduced by the hungarian mathematician Haar [19] in 1910, the other was published by the german Rademacher [31] in 1922, and the third one is due to Walsh [37] in 1923. All of these systems consist of piecewise constant functions. This is why they may have seemed to be rather artificial compared to the classical trigonometric, Legendre and Sturm-Liouville systems.

Haar proved in his doctoral dissertation in 1909 that, with respect to the system named after him, the Fourier series of every continuous function is uniformly convergent. This result was especially interesting since it was known by Du Bois Reymond that the trigonometric system fails to have this property. Moreover, as we know it now there is no uniformly bounded orthonormal system with the above property [27]. Walsh knew the results of Haar and he used them in his paper. On the other hand he didn't, maybe couldn't know about Rademacher's paper of 1922 when he handed his work to the American Mathematical Society in February 25., 1922. Since the system introduced by Walsh contains the one of Rademacher as a subsystem we may say that Walsh rediscovered it. However there was no hint about the connection between the two systems. That connection and its importance was observed by Paley [28] in 1932.

THE WALSH SYSTEM

The Walsh functions take on only the values $+1$ and -1 except for the points of $\mathbb{Q}$ which is the set of the so called dyadic rationals, i.e. $\mathbb{Q} := \{p2^{-q} : 0 \leq p \leq 2^n, p, q \in \mathbb{N}\}$ ($\mathbb{N} := \{0, 1, 2, \dots\}$). This feature makes them essentially different from the trigonometric functions. However, as it is already mentioned in the introductory part of Walsh's paper they imitate some properties of the *sine* and *cosine* systems. For instance, the n-th Walsh function changes its sign n times in $[0, 1)$, it is either odd or even with respect to the midpoint of $[0, 1)$, and they form a uniformly bounded orthonormal system. Walsh was probably motivated by these properties in creating

From left to right, Zbigniew Ciesielski, Institut Matematyczny PAN, Sopot, Poland, Ferenc Weisz, Eötvös L. University, Budapest, Hungary, Sh. Yano, Tohoku University, Japan, Ferenc Schipp, Eötvös L. University, Budapest, Hungary, Keith Phillips, University of Colarado, Colorado Springs, USA, William R. Wade, University of Tennessee, Knoxville, USA.

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